May 2002 #1 (QM)

$H(t) = \frac{1}{2}(p^2 + x^2) - \sqrt{2} f(t) x \qquad (m=1,\ \omega_0 = 1,\ \hbar = 1)$

$f(t)$ is a c-number function of time (commuting)

$|\psi(t)\rangle_S = U(t)|\psi(0)\rangle$

$|\psi\rangle_H = U^\dagger(t)|\psi(t)\rangle = |\psi(0)\rangle$

$O_H = U^\dagger(t) O_S U(t)$

$i\hbar \frac{d}{dt}|\psi(t)\rangle_S = H_S |\psi(t)\rangle_S$ Sch-Eq, true for arbitrary $|\psi(0)\rangle$

$$i\hbar \frac{dU}{dt} = H_S U \quad \Rightarrow \quad -i\hbar \frac{dU^\dagger}{dt} = U^\dagger H_S$$

$$i\hbar \frac{dO_H}{dt} = i\hbar \frac{d}{dt}\left(U^\dagger O_S U\right) = i\hbar\left[\frac{dU^\dagger}{dt} O_S U + U^\dagger O_S \frac{dU}{dt}\right]$$

$$i\hbar \frac{dO_H}{dt} = U^\dagger O_S H_S U - U^\dagger H_S O_S U = U^\dagger O_S U U^\dagger H_S U - U^\dagger H_S U U^\dagger O_S U$$

$$= O_H H_H - H_H O_H = [O_H, H_H]$$

$$H_H = U^\dagger H_S U = U^\dagger\left[\frac{1}{2}(p^2 + x^2) - \sqrt{2} f(t) x\right]U = \frac{1}{2}(p_H^2 + x_H^2) - \sqrt{2} f(t) x_H$$

because $U^\dagger x^2 U = U^\dagger x U U^\dagger x U = x_H x_H = x_H^2$

$O_H = x_H$:

$$i\hbar \frac{dx_H}{dt} = x_H\left[\frac{1}{2}(p_H^2 + x_H^2) - \sqrt{2} f(t) x_H\right] - \left[\frac{1}{2}(p_H^2 + x_H^2) - \sqrt{2} f(t) x_H\right] x_H$$

$$= \frac{1}{2}\left(x_H p_H^2 - p_H^2 x_H\right)$$

$$[x_H, p_H] = x_H p_H - p_H x_H = U^\dagger x U U^\dagger p U - U^\dagger p U U^\dagger x U = U^\dagger x p U - U^\dagger p x U$$

$$= U^\dagger (xp - px) U = U^\dagger (i\hbar I) U = i\hbar I = [x, p]$$

$$x_H p_H p_H - p_H p_H x_H = x_H p_H p_H - p_H(x_H p_H - i\hbar) = x_H p_H p_H - p_H x_H p_H + i\hbar p_H$$

$$= [x_H, p_H] p_H + i\hbar p_H = 2 i\hbar p_H$$

$$i\hbar \frac{dx_H}{dt} = i\hbar p_H$$

$$\boxed{\frac{dx_H}{dt} = p_H}$$

$$i\hbar\frac{dp_H}{dt} = p_H\left[\frac{1}{2}(p_H^2+x_H^2)-\sqrt{2}f(t)x_H\right]-\left[\frac{1}{2}(p_H^2+x_H^2)-\sqrt{2}f(t)x_H\right]p_H$$

$$=\frac{1}{2}(p_H x_H^2 - x_H^2 p_H) + \sqrt{2}f(t)(x_H p_H - p_H x_H)$$

$$=\frac{1}{2}(p_H x_H x_H - x_H(p_H x_H + i\hbar)) + \sqrt{2}f(t)i\hbar$$

$$=\frac{1}{2}(p_H x_H x_H - x_H p_H x_H - i\hbar x_H) + \sqrt{2}f(t)i\hbar$$

$$=\frac{1}{2}([p_H, x_H]x_H - i\hbar x_H) + \sqrt{2}f(t)i\hbar$$

$$= -i\hbar x_H + \sqrt{2}f(t)i\hbar$$

$$\boxed{\frac{dp_H}{dt} = -x_H + \sqrt{2}f(t)}$$

b. $f(t) = \begin{cases} f_0\cos\omega t & 0\le t\le T \\ 0 & \text{otherwise}\end{cases}$

Solving for $\omega=1$ (because that is used in (c))

$\dot{x}_H = p_H$

$\dot{p}_H = -x_H + \sqrt{2}f_0\cos t$

$\ddot{x}_H = \dot{p}_H = -x_H + \sqrt{2}f_0\cos t$

$\ddot{x}_H + \dot{x}_H = \sqrt{2}f_0\cos t$

Homogeneous solution: $x_c = A\cos t + B\sin t$

Particular solution: try $x_p = Ct\sin t$ $\quad \dot{x}_p = C\sin t + Ct\cos t$

$\ddot{x}_p = C\cos t + C\cos t - Ct\sin t$

$x_p + \ddot{x}_p = 2C\cos t = \sqrt{2}f_0\cos t \qquad C = \frac{f_0}{\sqrt{2}}$

$x_p = \frac{f_0}{\sqrt{2}}t\sin t$

$\Rightarrow \quad x_H = A\cos t + B\sin t + \frac{f_0}{\sqrt{2}}t\sin t$

$\dot{x}_H = p_H = B\cos t - A\sin t + \frac{f_0}{\sqrt{2}}\sin t + \frac{f_0}{\sqrt{2}}t\cos t$

initial conditions: $x_H = U^\dagger(t)\,x\,U(t) \qquad x_H(0) = x, \quad p_H(0) = p$

$x_H(0) = x = A$

$p_H(0) = B = p$

$\Rightarrow \quad x_H = x\cos t + (p + \frac{f_0}{\sqrt{2}}t)\sin t$

$p_H = (p + \frac{f_0}{\sqrt{2}}t)\cos t + (-x + \frac{f_0}{\sqrt{2}})\sin t$

c. $|\psi(0)\rangle = |0\rangle$, expectation value of energy gained at T

$$\langle\psi(t)|H(t)|\psi(t)\rangle = \langle\psi(0)|U^+H(t)U|\psi(0)\rangle$$

$$= \langle 0|H_H(t)|0\rangle = \langle E(t)\rangle$$

$$\langle E(t)\rangle = \langle 0|\tfrac{1}{2}(p_H^2+x_H^2) - \sqrt{2}f(t)x_H|0\rangle$$

for $t \le T$,

$$p_H^2 = \left(p+\tfrac{f_0}{\sqrt{2}}t\right)^2\cos^2 t + \left(-x+\tfrac{f_0}{\sqrt{2}}\right)^2\sin^2 t + 2\left(p+\tfrac{f_0}{\sqrt{2}}t\right)\left(-x+\tfrac{f_0}{\sqrt{2}}\right)\cos t\sin t$$

$$x_H^2 = x^2\cos^2 t + \left(p+\tfrac{f_0}{\sqrt{2}}t\right)^2\sin^2 t + 2x\left(p+\tfrac{f_0}{\sqrt{2}}t\right)\cos t\sin t$$

$$\frac{1}{2}(p_H^2+x_H^2) = \frac{1}{2}\left[\left(p+\frac{f_0}{\sqrt{2}}t\right)^2 + x^2 - \sqrt{2}f_0 x\sin^2 t + \frac{f_0^2}{2}\sin^2 t + \sqrt{2}f_0\left(p+\frac{f_0}{\sqrt{2}}t\right)\cos t\sin t\right]$$

$$-\sqrt{2}f(t)x_H = -\sqrt{2}f_0\cos t\left[x\cos t + \left(p+\tfrac{f_0}{\sqrt{2}}t\right)\sin t\right]$$

$$x = \frac{x_0}{\sqrt{2}}(a+a^+) \longrightarrow \frac{1}{\sqrt{2}}(a+a^+) \qquad x^2 = \frac{1}{2}(a^2+aa^++a^+a+a^{+2})$$

$$p = \frac{\hbar(a-a^+)}{i\sqrt{2}x_0} \longrightarrow \frac{1}{i\sqrt{2}}(a-a^+) \qquad p^2 = -\frac{1}{2}(a^2-aa^+-a^+a+a^{+2})$$

$$\langle 0|x|0\rangle = 0 \qquad \langle 0|p|0\rangle = 0$$

$$\langle 0|x^2|0\rangle = \tfrac{1}{2}\langle 0|aa^+|0\rangle = \tfrac{1}{2}$$

$$\langle 0|p^2|0\rangle = \tfrac{1}{2}\langle 0|aa^+|0\rangle = \tfrac{1}{2}$$

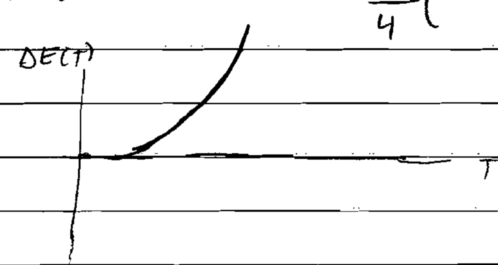
$$\langle E(t)\rangle = \frac{1}{2}\left[\frac{1}{2} + \frac{f_0^2t^2}{2} + \frac{1}{2} + \frac{f_0^2\sin^2 t}{2} + f_0^2 t\cos t\sin t\right] - f_0^2 t\cos t\sin t$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{f_0^2t^2}{4} + \frac{f_0^2\sin^2 t}{4} - \frac{f_0^2 t\cos t\sin t}{2}$$

$$= \frac{1}{2} + \frac{f_0^2}{4}\left(t^2 + \sin^2 t - 2t\cos t\sin t\right)$$

ground state energy, $\frac{1}{2}\hbar\omega$

Energy gain $\langle\Delta E(T)\rangle = \dfrac{f_0^2}{4}\left(T^2 + \sin^2 T - 2T\cos T\sin T\right)$



(goes as T^4 for small T)

then T^2 for large T