2011 - Part I - Question 1B
Applied Mathematics (Quickie)

May 10, 2011

We want to find the asymptotic solutions to

$$\psi'' + Q(x)\psi = 0 \tag{1}$$

This is just the standard WKB problem. As the problem instructs, we look for solutions of the form $\psi = e^S$. Plugging this solution into Eq. 1 and find

$$S'' + (S')^2 + Q = 0 \tag{2}$$

Now, we have no information about Q. Thus, we can only guess a balance and find the condition such that the balance holds. $(S')^2$ *usually* dominates over S'', so we choose a balance

$$(S')^2 = -Q$$
$$S' = \pm i Q^{\frac{1}{2}} \tag{3}$$

In order for this to be a valid balance, we require

$$|S''| << \left|(S')^2\right|$$
$$\left|\frac{Q'}{Q^{\frac{1}{2}}}\right| << |Q|$$
$$\left|\frac{Q'}{Q^{\frac{3}{2}}}\right| << 1 \tag{4}$$

This is the standard limit for the WKB approximation.

Now we want the second order solution, so we add a g' to S' and plug it back into Eq. 2 to find

$$\pm\frac{i}{2}\frac{Q'}{Q^{\frac{1}{2}}} + g'' - \not{Q} \pm 2iQ^{\frac{1}{2}}g' + (g')^2 + \not{Q} = 0 \tag{5}$$

Balancing the two $\pm$ terms, we find

$$g' = -\frac{1}{4}\frac{Q'}{Q} \tag{6}$$

Now we need to ensure that *this* is a legitimate balance too. Thus, we compare $(g')^2$ to the balanced terms.

$$(g')^2 << \frac{Q'}{Q^{\frac{1}{2}}}$$

$$\frac{(Q')^2}{Q^2} << \frac{Q'}{Q^{\frac{1}{2}}}$$

$$\frac{Q'}{Q^{\frac{3}{2}}} << 1$$

Again, we find the same condition we imposed for first order balance. Next, we compare g''

$$g'' << \frac{Q'}{Q^{\frac{1}{2}}}$$

$$\frac{(Q')^2}{Q^2} + \frac{Q''}{Q} << \frac{Q'}{Q^{\frac{1}{2}}}$$

The first term on the LHS is small by the WKB limit. The second term places a new limit on Q''. In general we ignore this, perhaps because a large Q'' would cause a large Q' someplace over the domain, resulting in a violation of the WKB limit.

Putting these results together, we find our second order solution for Eq. 1

$$S' = -\frac{1}{4}\frac{Q'}{Q} \pm iQ^{\frac{1}{2}}$$

$$S = -\frac{1}{4}\ln Q \pm i\int_{x_0}^{x} Q^{\frac{1}{2}}dx$$

$$\psi = e^S = \frac{1}{Q^{\frac{1}{4}}}\exp\left(\pm i\int_{x_0}^{x} Q^{\frac{1}{2}}dx\right) \tag{7}$$

This is the standard WKB solution.

Lastly, we are asked for values of Q for which these solutions will fail. Clearly, $Q = 0$ is one. Another one would be where the WKB limit breaks down

$$Q' = Q^3/2 \Rightarrow Q \sim \frac{1}{(x+c)^2}$$

In fact, one can see that any Q with the form

$$Q \sim \frac{1}{(x+c)^n} \quad n \geq 2$$

will violate the WKB limit. Certainly there are many more Qs that would violate the WKB limit, but these are some simple ones to see.