Generals 2012: Part I Q2 Waves and Instabilities

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(a)

Relativistic Vlasov-Poisson for electrons:

$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - e\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$, $\mathbf{E} = -\nabla \phi$, $\nabla \cdot \mathbf{E} = 4\pi\rho$

Do the usual linearization and space/time Fourier transforms (sufficient) to get the dispersion relation

Linearized: $\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} - e\mathbf{E}_1 \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0$

Fourier transformed:

$-i\omega \hat{f}_1 + \mathbf{v} \cdot i\mathbf{k}\hat{f}_1 + ei\hat{\phi}_1\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0$, $-(ik)^2 \hat{\phi}_1 = 4\pi\hat{\rho}_1$

$-\omega \hat{f}_1 + vk\cos\theta \hat{f}_1 + e\hat{\phi}_1\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0$

$\hat{f}_1 (vk\cos\theta - \omega) = -e\hat{\phi}_1\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}}$

$\hat{f}_1 = \frac{-e\hat{\phi}_1\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}}}{(vk\cos\theta - \omega)}$

The Poisson equation gives:

$\hat{\phi}_1 = 4\pi\frac{\hat{\rho}_1}{k^2}$

The perturbed charge density is (I'm not showing Jacobian factors):

$\rho_1(\mathbf{x}) = -e\int dpd\theta d\phi f_1$

so

$\hat{\rho_1 (\mathbf{x})} = -e\int dpd\theta d\phi \hat{f}_1$

Plugging in for $\hat{\rho_1}$,$\hat{f}_1$ in the Poisson equation:

$\hat{\phi}_1 = 4\pi\frac{\hat{\rho}_1}{k^2} = 4\pi\frac{e^2}{k^2}\int dpd\theta d\phi \frac{\hat{\phi}_1\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}}}{(vk\cos\theta - \omega)}$

So:

$\epsilon(\omega, k) = 1 - \frac{4\pi e^2}{k^2}\int dpd\theta d\phi \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}}}{(vk\cos\theta - \omega)}$

Given $f_0(\mathbf{p}) = f_0(p) = \frac{n_0 c^3}{8\pi T^3}\exp\left(-\frac{cp}{T}\right)$

$\epsilon(\omega, k) = 1 - \frac{4\pi e^2}{k^2}\int dpd\theta d\phi \frac{k\cos\theta\left(-\frac{c}{T}\right)f_0(p)}{(vk\cos\theta - \omega)}$

With the Jacobian factors:

$\epsilon(\omega, k) = 1 + \frac{4\pi e^2}{k^2 T}\int_0^\infty dp 2\pi p^2 f_0(p)\int_0^\pi \frac{ck\cos\theta}{(vk\cos\theta - \omega)}\sin\theta d\theta$

Or, in terms of $u = \frac{\omega}{ck}$

$\epsilon(\omega, k) = 1 + \frac{4\pi e^2}{k^2 T}\int_0^\infty dp 2\pi p^2 f_0(p)\int_0^\pi \frac{\cos\theta}{\left(\frac{v}{c}\cos\theta - u\right)}\sin\theta d\theta$

Since the plasma is ultrarelativistic $\frac{v}{c} \to 1$, and (a) is complete

(b)

First simplifying the expression for $\epsilon(\omega,k)$ using the given integral $\int_0^\infty 4\pi p^2 f_0(p)\,dp = n_0$ and $\frac{1}{\lambda_D^2} = \frac{4\pi n_0 e^2}{T}$:

$\epsilon(\omega,k) = 1 + \frac{1}{2k^2\lambda_D^2}\int_0^\pi \frac{\cos\theta}{(\cos\theta - u)}\sin\theta d\theta$

Using $s = \cos\theta$ as suggested:

$\epsilon(\omega,k) = 1 + \frac{1}{2k^2\lambda_D^2}\int_{-1}^1 \frac{s}{(s-u)}ds$

Rewriting (using the given identity):

$\epsilon(\omega,k) = 1 + \frac{1}{2k^2\lambda_D^2}\int_{-1}^1 \left(1 + \frac{1}{\frac{s}{u}-1}\right)ds$

$\epsilon(\omega,k) = 1 + \frac{1}{2k^2\lambda_D^2}\left(2 + u\int_{-1}^1 \frac{1}{s-u}ds\right)$

The integral has a pole if $-1 < u < 1$, we understand the divergent integral through the Landau contour.

For $-1 < u < 1$ take the principle value and the residue

For $|u| > 1$ residue $\rightarrow 0$, just principle value

For $k > 0$, the residue is $i\pi$ (for $k < 0$, $-i\pi$)

This gives $\epsilon''(\omega,k){=}\frac{\pi|u|}{2k^2\lambda_D^2}$, $-1 < u < 1$

Now, I'll do the principle value to find $\epsilon'(\omega,k)$:

$\int_{-1}^1 \frac{1}{s-u}ds = \int_{-1}^{u-\epsilon}\frac{1}{s-u}ds + \int_{u+\epsilon}^1 \frac{1}{s-u}ds$

$\ln|s-u|]_{-1}^{u-\epsilon} + \ln|s-u|]_{u+\epsilon}^1$

$\ln|-\epsilon| - \ln|-1-u| + \ln|1-u| - \ln|\epsilon|$

So the principle value is:

$\ln\left|\frac{1-u}{1+u}\right|$

So in total I find $\epsilon'(\omega,k) = 1 + \frac{1}{k^2\lambda_D^2} + \frac{u}{2k^2\lambda_D^2}\ln\left|\frac{1-u}{1+u}\right|$

There is a logarithmic singularity in ϵ' at $u = 1$

(c)

The sketch for $\epsilon''(\omega,k){=}\frac{\pi|u|}{2k^2\lambda_D^2}$, for $-1 < u < 1$ (and 0 for $|u| > 1$) is straightforward once one has the result.

For $\epsilon'(\omega,k)$ I first find the asymptotic values as suggested.

If $u \ll 1$ then $\epsilon'(\omega,k) \rightarrow 1 + \frac{1}{k^2\lambda_D^2}$, this is Debye shielding

If $u \gg 1$ then the logarithm can be rewritten as:

$\ln\left|\frac{1-u}{1+u}\right| = \ln\left|\frac{\frac{1}{u}-1}{\frac{1}{u}+1}\right| \approx \ln\left|\left(\frac{1}{u}-1\right)\left(1-\frac{1}{u}\right)\right| = \ln\left|-1+\frac{2}{u}-\frac{1}{u^2}\right| \approx -\frac{2}{u}$

Then $\epsilon'(\omega,k) \rightarrow 1$, indicating the wave doesn't feel the effect of the particles.

We know $\epsilon'(\omega,k)$ for $u \ll 1$ and $u \gg 1$, and that at $u = 1$ there is a logarithmic singularity.

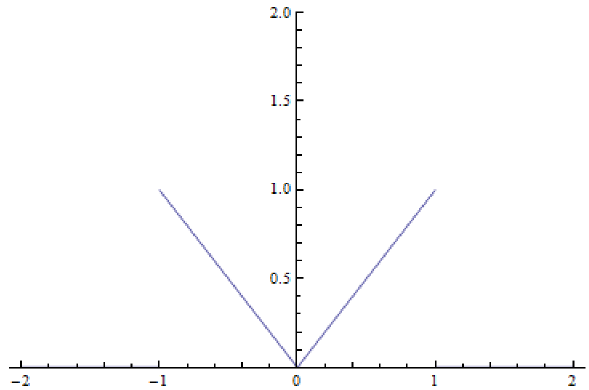

Figure 1: Plot of $\epsilon''(\omega,k)/\left(\frac{\pi}{2k^2\lambda_D^2}\right)$ versus u

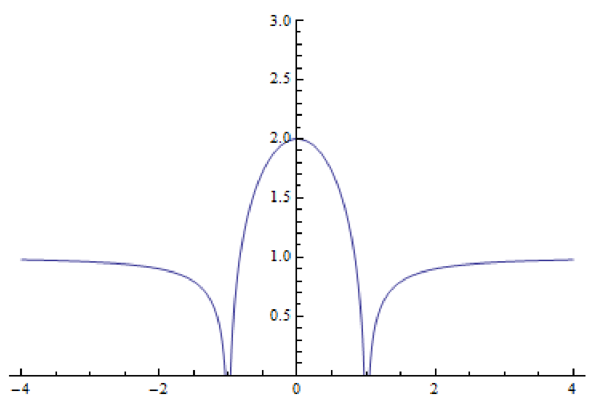

Figure 2: Plot of $\epsilon'(\omega,k)$ versus u with $k^2\lambda_D^2$ set to 1