General Examination May 9, 2011 Part I

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4 Neoclassical Plasma Physics [60 Points]

Part 1: Estimate the terms in the "neoclassical" two-by-two matrix that relates the fluxes –trapped-particle flux and charge flux (i.e. current density) to the driving forces (density gradient and parallel electric field) in the "banana regime".

[25 pts.] Give the trapped-particle flux using simplified (heuristic) estimates of the coefficients for the driving forces.

As stated in the text of the problem, there are two driving forces of interest. We will look at the forces from a density gradient and toroidal electric field.

A density gradient will result in random walk diffusion of particles. In a classical analysis we would quickly write a diffusion coefficient $D \sim \rho^2\nu$. In neoclassical analysis, the diffusion of interest is the much quicker banana diffusion. In this case, we start with the basic relations

$$\langle\Gamma\rangle = -D_b\frac{\partial n}{\partial r}$$

$$D_b \sim \frac{\langle\Delta x\rangle}{\Delta t} \sim \Lambda^2\nu_{eff}f_T$$

Where r is the radial coordinate (banana are centered around the outboard midplane for a tokamak, so this is effectively the major radius coordinate R in this case), D_b is the banana diffusion coefficient, Λ is the banana width, ν_{eff} is the effective collision frequency, and f_T is the fraction of trapped particles.

To find Λ

$$\Lambda = v_{drift}\tau_{bounce}$$

Using the curvature drift

$$v_{drift} = -\frac{\mu\nabla B\times B}{eB} \sim \frac{v_T^2}{\Omega R}$$

And

$$\tau_{bounce} = \int_{-\theta_b}^{\theta_b}\frac{ds}{|v_\parallel|} \sim \frac{Rq}{\epsilon^{1/2}v_T}$$

Finally

$$\Lambda \sim \frac{v_T^2}{\Omega R}\frac{Rq}{\epsilon^{1/2}v_T} \sim \frac{\rho q}{\epsilon^{1/2}}$$

Now we recal the trapping condition

$$w_{\perp 0} + w_{\parallel 0} = \mu B_{max}$$

$$\frac{w_{\parallel 0}}{w_{\perp 0}} = \frac{1+\epsilon}{1-\epsilon} - 1$$

$$v_{\parallel} \sim \epsilon^{1/2} v_T$$

The relevant collisional frequency is the frequency at which particles change their velocity enough to leave their banana orbit $\nu_{eff} \sim \nu^{90}\epsilon^{-1}$. Additionally, $f_T \sim \epsilon^{1/2}$. Putting it all together

$$D_b \sim \left(\frac{\rho q}{\epsilon^{1/2}}\right)^2 \nu^{90}\epsilon^{-1/2} \sim \frac{\rho^2 q^2 \nu^{90}}{\epsilon^{3/2}}$$

$$\langle \Gamma \rangle \sim -\frac{\rho^2 q^2 \nu^{90}}{\epsilon^{3/2}} \frac{\partial n}{\partial r}$$

A toroidal electric field -providing the second driving force- will result in the classic ware pinch. This is a radial drift of banana orbits.

$$v_{ware} = \langle v_r \rangle \sim v_{drift} \Delta\theta_{bounce} \sim v_{drift} \frac{\Delta v_{\parallel}}{v_{\parallel}}$$

The small difference in parallel velocity between the two halves of the orbit (one going with the E-field and one against) can be estimated as follows

$$\Delta v_{\parallel} = \frac{eE_{\parallel}}{m} \tau_{bounce} \sim \frac{eE_{\parallel}}{m} \frac{Rq}{v_T \epsilon^{1/2}}$$

Using what we already know about the velocity scalings

$$v_{ware} \sim v_d \frac{\Delta v_{\parallel}}{v_{\parallel}} \sim \frac{v_T^2}{\Omega R} \frac{eE_{\parallel}}{m} \frac{Rq}{\epsilon^{1/2} v_T} \frac{1}{\epsilon^{1/2} v_T} \sim \frac{eE_{\parallel}}{\Omega m/q} \sim \frac{cE_{\parallel}}{B_p}$$

The corresponding particle flux (averaged over a banana orbit) is

$$\langle \Gamma \rangle = -f_T n v_{ware} \sim -\frac{c\epsilon^{1/2}}{B_p} n E_{\parallel}$$

[15 pts.] Give a simplified (heuristic) estimate for the neoclassical current density by specifying the coefficients for the driving forces. The density gradient results in a bootstrap current.

$$J_b = env f_{c-c}$$

where e is the particles charge, n is the particle density, v is the current carrying particles velocity in the toroidal direction, and $f_{c-c}$is the fraction of co-moving vs counter-moving particles. The key point to not is that the passing and trapped particle populations are in collisional equilibrium (with particles constantly changing from one to the other through pitch angle scattering). Thus, the fraction f_{c-c} of the passing population is equal to the corresponding fraction for the trapped population

$$f_{c-c} \sim -\frac{\Lambda}{n} \frac{\partial n}{\partial r}$$

Where we assumed the density gradient was negative and explicitly made the fraction positive. The current carrying particles are the untrapped population, so we estimate $v \sim v_T$. Plugging in

$$J_b \sim env_T \frac{Rq}{\epsilon^{1/2} v_T} \frac{\partial n}{\partial r} \sim -\frac{c\epsilon^{1/2}}{B_p} T \frac{\partial n}{\partial r}$$

There is also a current driven by the toroidal electric field. classically we would just write $J_s = \sigma_\parallel E_\parallel$. Taking the trapped particles into account

$$J \sim \sigma_\parallel E_\parallel (1 - f_T) \sim J_s \left(1 - \epsilon^{1/2}\right)$$

We are just looking for the "neoclassical current", so we write

$$\langle J - J_s \rangle = -\epsilon^{1/2} \sigma_\parallel E_\parallel$$

Putting everything into matrix form

$$\begin{bmatrix} \langle \Gamma \rangle \\ \langle J - J_s \rangle \end{bmatrix} = \begin{bmatrix} \frac{\rho^2 q^2 \nu^{90}}{T \epsilon^{3/2}} & \frac{c\epsilon^{1/2}}{B_p} \\ \frac{c\epsilon^{1/2}}{B_p} & \frac{\epsilon^{1/2} \sigma_\parallel}{n} \end{bmatrix} \begin{bmatrix} -T \frac{\partial n}{\partial r} \\ -nE_\parallel \end{bmatrix}$$

Note the Onsager symmetry between the off-diagonal coefficient matrix elements.

Part 2: Electrostatic drift waves (and associated instabilities) are often invoked when neoclassical theory prove inadequate to account for the higher levels of transport often observed in toroidal experiments.

[15 pts.] Estimate the perturbed density responses for kinetic "adiabatic" electrons and the cold fluid ions respectively. Combine to give the drift-wave dispersion relation.

We write down the adiabatic electron response (taylor expand the gibbs distribution)

$$\frac{\delta n_e}{n_e} \approx \frac{e\phi}{T}$$

For the ion response, we use the fluid equations. Start with the continuity equation

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}) = 0$$

Lets assume a density gradient in the x-direction and define $L = \left[-\frac{1}{n}\frac{\partial n}{\partial x}\right]^{-1}$. Linearizing, assuming no mean flow, we get

$$\frac{\partial \delta n_i}{\partial t} - v_x \frac{n}{L} + n \nabla \cdot \vec{v} = 0$$

In Fourier space

$$\frac{-i\omega \delta n_i}{n_i} - \frac{v_x}{L} + i\vec{k} \cdot \vec{v} = 0$$

Using the momentum equation with $T_i = 0$, $\vec{k} = k_y \hat{y}$ and $\vec{B} = B\hat{z}$

$$\frac{\partial}{\partial t}(m_i n_i \vec{v}) - en_i \vec{E} - \frac{en_i}{c} \vec{v} \times \vec{B} = 0$$

$$-i\omega\vec{v} + \frac{eik_y\phi}{m_i}\hat{y} - \Omega_i\vec{v}\times\hat{z} = 0$$

Solving by components and dropping terms of order $(\omega/\Omega_i)^2$

$$v_x \approx -i\frac{e}{m_i\Omega_i}k_y\phi$$

$$v_y \approx -\frac{e\omega k_y\phi}{m_i\Omega_i^2}$$

At this point we can recognize the physics that is fundamentally responsible for our drift wave motion: $\vec{E}\times\vec{B}$ drift in the y-direction and polarization drift in the x-direction. Plugging back into the continuity equation

$$\frac{\delta n_i}{n_i} = \frac{e\phi}{T_e}\left(\frac{k_yT_e}{m_i\Omega_i L}\frac{1}{\omega} - \frac{k_y^2T_e}{m_i\Omega_i^2}\right) = 0$$

Demanding neutrality

$$\frac{\delta n_i}{n_i} = \frac{\delta n_e}{n_e}$$

$$1 - \frac{k_yT_e}{m_i\Omega_i L}\frac{1}{\omega} + \frac{k_y^2T_e}{m_i\Omega_i^2} = 0$$

This is cleaned up by defining $\omega_{*} = k_yT_e/m_i\Omega_i L$, $c_s^2 = T_e/m_i$, and $\rho_s = c_s/\Omega_i$

$$1 - \frac{\omega_{*}}{\omega} + k_y^2\rho_s^2 = 0$$

Finally, we rewrite this in the form of the classic drift wave dispersion relation

$$\omega = \frac{\omega_{*}}{1 + k_y^2\rho_s^2}$$

[5 pts.] Justify the use of the "quasineutrality condition" relating the perturbed ion and electron density responses.

The quasineutrality condition is a statement that the wavelengths of interest are much larger then a Debye length.

This is seen by looking at the Poisson equation

$$\nabla^2\phi = 4\pi en\left(\frac{\delta n_i}{n} - \frac{\delta n_e}{n}\right)$$

$$k^2\phi = \frac{4\pi ne^2}{T_e}\phi\left(\frac{\delta n_i}{\delta n_e} - 1\right)$$

$$\frac{\delta n_i}{\delta n_e} - 1 = k^2\lambda_{De}^2 \ll 1$$

$$\delta n_i \approx \delta n_e$$