2005 Part I Q1 — Asymptotics

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}y = 0$$

A. asymptotic solutions for $x \to 0$ $\qquad x^2y'' + y' + y = 0$

$$x^2[S'' + (S')^2] + S' + 1 = 0$$

$$x^2S' + 1 = 0 \qquad S' = -\frac{1}{x^2}$$

$$S' = -\frac{1}{x^2} + g' \qquad S'' = \frac{2}{x^3} + g'' \qquad (S')^2 = \frac{1}{x^4} - \frac{2g'}{x^2} + (g')^2$$

$$x^2\left[\frac{2}{x^3} + \underbrace{\cancel{g''}}_{\sim 0} + \cancel{\frac{1}{x^4}} - \frac{2g'}{x^2} + (g')^2\right] - \cancel{\frac{1}{x^2}} + g' + \underbrace{\cancel{1}}_{\sim 0} = 0$$

$$\frac{2}{x} - g' + x^2g'^2 = 0 \qquad g' = \frac{2}{x}$$

$$S' = -\frac{1}{x^2} + \frac{2}{x} \qquad S = \frac{1}{x} + 2\ln x \qquad \boxed{y = x^2e^{\frac{1}{x}}}$$

$$S' = -1 \qquad \boxed{y = e^{-x}} \qquad y \to \text{constant as } x \to 0$$

B. For $x \to \infty$, $\qquad x^2[S'' + (S')^2] + S' + 1 = 0$

Attempt dominant balance: $(S')^2 = -\frac{1}{x^2}$ $\quad S' = \pm\frac{i}{x}$

But $S''$ is also of the same order

$\Rightarrow$ write $S' = \frac{c}{x}$ $\quad S'^2 = \frac{c^2}{x^2}$ $\quad S'' = -\frac{c}{x^2}$

$$\Rightarrow -c + c^2 + 1 = 0 \qquad c = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$S' = \frac{1}{2x} \pm i\frac{\sqrt{3}}{2x} \qquad S = \left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)\ln x$$

$$y = e^{\frac{1}{2}\ln x}\, e^{\pm i\frac{\sqrt{3}}{2}\ln x} = \sqrt{x}\, e^{\pm i\frac{\sqrt{3}}{2}\ln x}$$

Real solutions: $$y = \begin{cases} \sqrt{x}\cos\left(\frac{\sqrt{3}}{2}\ln x\right) \\ \sqrt{x}\sin\left(\frac{\sqrt{3}}{2}\ln x\right) \end{cases}$$