2002 Part 1 Q2 Asymptotics

$$\frac{d^3y}{dx^3} + x^2 y - \sinh x = 0$$ asymptotic behavior, $x \to \infty$, for real y

Inhomogeneous term. First find the particular solution

$y_p = \frac{\sinh x}{x^2}$ Then $y''' \sim \frac{\cosh x}{x^2}$, $<< \sinh x$

Homogeneous solution $y''' + x^2 y = 0$

$y = e^S \rightarrow y' = S' e^S \longrightarrow y'' = [S'' + (S')^2] e^S$

$y''' = [S''' + S''S' + 2(S')S'' + (S')^3] e^S = [S''' + 3S''S' + (S')^3] e^S$

$$S''' + 3S''S' + (S')^3 + x^2 = 0$$

Try $(S')^3 + x^2 = 0$ $S'^3 = -x^2$ $S' = -x^{2/3}$, $e^{i\pi/3} x^{2/3}$, $e^{i5\pi/3} x^{2/3}$

$\rightarrow S'' \sim x^{-1/3}$, $S''' \sim x^{-4/3}$ good dominant balance ✓

$S' = (-1)^{1/3} x^{2/3} + g'$

$S'' = (-1)^{1/3} \frac{2}{3} x^{-1/3} + g''$

$S''' = (-1)^{1/3} \frac{-2}{9} x^{-4/3} + g'''$

$S'S'' = (-1)^{2/3} \frac{2}{3} x^{1/3} + (-1)^{1/3} x^{2/3} g'' + (-1)^{1/3} \frac{2}{3} x^{-1/3} g' + g'g''$

$(S')^3 = -x^2 + 3(-1)^{2/3} x^{4/3} g' + 3(-1)^{2/3} x^{2/3} (g')^2 + (g')^3$

$$(-1)^{1/3} \frac{-2}{9} x^{-4/3} + \underbrace{\cancel{g'''}}_{\sim 0} + 3\left[(-1)^{2/3} \frac{2}{3} x^{1/3} + \underbrace{\cancel{(-1)^{1/3} x^{2/3} g''}}_{\sim 0} + \underbrace{\cancel{(-1)^{1/3} \frac{2}{3} x^{-1/3} g'}}_{\sim 0} + \underbrace{\cancel{g'g''}}_{\sim 0}\right] - \cancel{x^2} + 3(-1)^{2/3} x^{4/3} g' + 3(-1)^{2/3} x^{2/3} g'^2$$
$$+ (g')^3 + \cancel{x^2} = 0$$

$$\Rightarrow (-1)^{1/3} \frac{-2}{9} x^{-4/3} + \underbrace{3(-1)^{2/3} \frac{2}{3} x^{1/3} + 3(-1)^{2/3} x^{4/3} g'}_{} + 3(-1)^{2/3} x^{2/3} g'^2 + (g')^3 = 0$$

$\frac{2}{3} x^{1/3} + x^{4/3} g' = 0$ $g' = -\frac{2}{3x}$

$S' = (-1)^{1/3} x^{2/3} - \frac{2}{3x}$

$S = (-1)^{1/3} \frac{3}{5} x^{5/3} - \frac{2}{3} \ln x$ $(-1)^{1/3} = -1, e^{i\pi/3}, e^{i5\pi/3}$

$y = e^S$ $\quad y = e^{(-1)^{1/3}\frac{3}{5}x^{5/3}} e^{-\frac{2}{3}\ln x}$ $\quad y = x^{-\frac{2}{3}} e^{(-1)^{1/3}\frac{3}{5}x^{5/3}}$

$$\boxed{y_1 = x^{-\frac{2}{3}} e^{-\frac{3}{5}x^{5/3}}}$$

$y_2 = x^{-\frac{2}{3}} e^{e^{i\pi/3}\frac{3}{5}x^{5/3}}$ $\quad y_3 = x^{-\frac{2}{3}} e^{e^{i\frac{5\pi}{3}}\frac{3}{5}x^{5/3}}$

$$e^{i\pi/3} = \cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3} = \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

$$e^{i5\pi/3} = \cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3} = \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}$$

$$y_{2,3} = x^{-\frac{2}{3}} e^{\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)\frac{3}{5}x^{5/3}} = x^{-\frac{2}{3}} e^{\frac{3}{10}x^{5/3}} e^{\pm i\frac{3\sqrt{3}}{10}x^{5/3}}$$

Real solutions:

$$\boxed{\begin{aligned} y_2 &= x^{-2/3} e^{\frac{3}{10}x^{5/3}} \cos\left(\frac{3\sqrt{3}}{10}x^{5/3}\right) \\ y_3 &= x^{-2/3} e^{\frac{3}{10}x^{5/3}} \sin\left(\frac{3\sqrt{3}}{10}x^{5/3}\right) \end{aligned}}$$

$$y \cong \frac{\sinh x}{x^2} + c_1 x^{-\frac{2}{3}} e^{-\frac{3}{5}x^{5/3}} + x^{-\frac{2}{3}} e^{\frac{3}{10}x^{5/3}}\left(c_2\cos\frac{3\sqrt{3}}{10}x^{5/3} + c_3\sin\frac{3\sqrt{3}}{10}x^{5/3}\right)$$