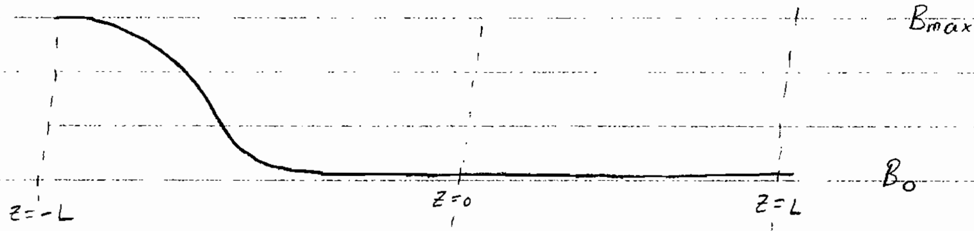
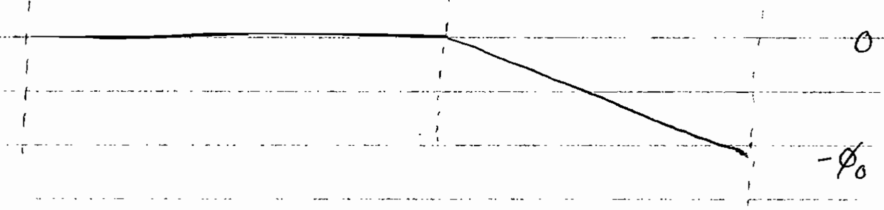
2002 Part I Q6

a. Magnetic field: $B(z)$



Electric potential $\phi(z)$



$$C_{pa}(f_e) = \nu(v)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} f_e \qquad \text{where } \mu = \frac{v_z}{v} \quad v = |\vec{v}|, \quad \nu(v) \sim v^{-3}$$

Suppose all collisions occur at $z=0$

a. In the absence of collisions, conditions for electron trapping

$$E = \tfrac{1}{2}mv_{\parallel}^2 + \tfrac{1}{2}mv_{\perp}^2 - e\phi(z) = \tfrac{1}{2}mv_{\parallel 0}^2 + \tfrac{1}{2}mv_{\perp 0}^2 \qquad \mu B = \tfrac{1}{2}mv_{\perp}^2 \Rightarrow \mu = \frac{\tfrac{1}{2}mv_{\perp 0}^2}{B_o}$$

$$v_{\parallel}^2 + v_{\perp 0}^2\frac{B}{B_0} - \frac{2e\phi(z)}{m} = v_{\parallel 0}^2 + v_{\perp 0}^2$$

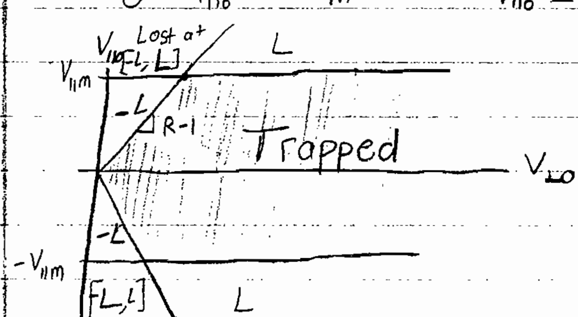
$$v_{\parallel}^2 = v_{\parallel 0}^2 - v_{\perp 0}^2\left(\frac{B}{B_0} - 1\right) + \frac{2e\phi(z)}{m}$$

Condition for trapping for $z<0$: repelled by the mirror by $z=-L$ $(\phi=0)$

$$0 = v_{\parallel 0}^2 - v_{\perp 0}^2(R-1) \Rightarrow \frac{v_{\parallel 0}^2}{v_{\perp 0}^2} \le R-1 \qquad \left|\frac{v_{\parallel 0}}{v_{\perp 0}}\right| \le \sqrt{R-1}$$

Condition for trapping for $z>0$: electrically repelled by $z=L$ $(\phi = -\phi_o)$

$$0 = v_{\parallel 0}^2 - \frac{2e\phi_0}{m} \qquad v_{\parallel 0}^2 \le \frac{2e\phi_0}{m} \qquad |v_{\parallel 0}| \le \sqrt{\frac{2e\phi_0}{m}} \equiv v_{\parallel m}$$



b. For electrons leaving at $z=-L$: $\left|\frac{v_{\parallel 0}}{v_{\perp 0}}\right| \geq R-1$

at $z=L$: $|v_{\parallel 0}| \geq \sqrt{\frac{2e\phi_0}{m}} = v_{\parallel m}$

Infrequent pitch angle scattering (conserves energy)

The side electrons are lost depends only on its total energy. If the energy is less than the energy at the intersection of the trapping boundaries, the electron scatters into the magnetically lost region. If the energy is greater than this critical energy, then the electron scatters into the electrically lost region. [This picture is for scattering occurring at the midplane]

$$E_{crit} = \tfrac{1}{2}mv_{\parallel m}^2 + \tfrac{1}{2}mv_{\parallel m}^2\left(\frac{1}{R-1}\right) = e\phi_0\left(1+\frac{1}{R-1}\right) = e\phi_0\left(\frac{R}{R-1}\right)$$

$$\frac{1}{2}mv_{crit}^2 = e\phi_0\frac{R}{R-1} \Rightarrow v_c = \sqrt{\frac{2e\phi_0}{m}}\sqrt{\frac{R}{R-1}}$$

If $v<v_c$, electron is lost at $z=-L$

If $v>v_c$, electron is lost at $z=L$

c. If the bounce frequency is much larger than the collision frequency, then knowing $f(v,\mu,t)$ at the midplane effectively tells you the distribution function everywhere in space. Knowing this is sufficient for confinement purposes.

We may treat collisions as only occuring at the midplane

- for convenience?
- $\boxed{\text{because collisions are small-angle?}}$ → a collision which changes the pitch angle by $\delta\theta$ (very small) not at the midplane is {almost} equivalent to a change $\delta\theta$ at the midplane. [But this isn't quite true]
- Note that since $v_\parallel$ decreases as $z \to L$, v decreases and collisions are more frequent near $z=L$

d. Since we only worry about f at the midplane, and changes due to collisions, we don't need the $\vec{v}\cdot\nabla f$ or $\frac{\vec{F}}{m}\cdot\frac{\partial f}{\partial \vec{v}}$ terms: (we already know how they affect the distribution function)

$$\frac{\partial f}{\partial t} = C(f) + S(v,\mu,t)$$

Boundary conditions: $f=0$ at the loss cone boundary

$\Rightarrow$ For $v<v_c$, the condition is $f=0$ when $\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} = R-1$

$$\frac{v_{\parallel 0}^2}{v^2 - v_{\parallel 0}^2} = R-1 \qquad \frac{v_{\parallel 0}^2/v^2}{1-v_{\parallel 0}^2/v^2} = R-1 \Rightarrow \frac{\mu^2}{1-\mu^2} = R-1$$

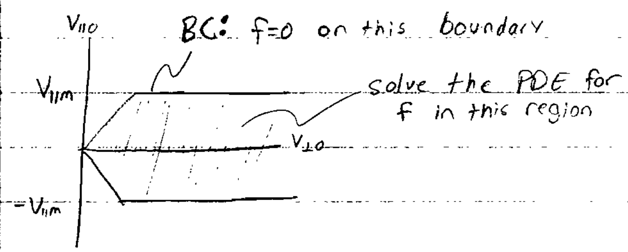
$$\mu^2 = (R-1)(1-\mu^2) \rightarrow \mu^2 + \mu^2 R - \mu^2 = R-1 \qquad \mu^2 = \frac{R-1}{R} = 1-\frac{1}{R} \equiv \mu_m^2$$

$$\boxed{\text{For } v<v_c, \quad f(v, \mu = \pm\mu_m, t) = 0}$$

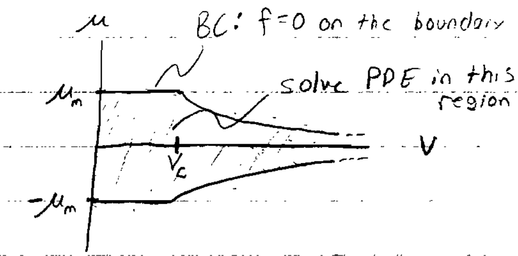
$\Rightarrow$ For $v>v_c$, the condition is $f=0$ when $v_\parallel^2 = v_{\parallel m}^2$

$$\frac{v_\parallel^2}{v^2}v^2 = v_{\parallel m}^2 \Rightarrow \mu^2 v^2 = v_{\parallel m}^2 \qquad \mu^2 = \frac{v_{\parallel m}^2}{v^2}$$

$$\boxed{\text{For } v>v_c, \quad f\left(v, \mu = \pm\frac{v_{\parallel m}}{v}, t\right) = 0}$$



$v_{\parallel 0} - v_{\perp 0}$ space



$\mu - v$ space

e. $S(v,\mu,t) \rightarrow S(v)$.

Solve for f in steady state: $\frac{\partial f}{\partial t} = 0$

$$\nu \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} f = -S$$

- Since there are no v derivatives, $\nu(v)$ and $S(v)$ act like constants.

$$\Rightarrow (1-\mu^2)\frac{\partial f}{\partial \mu} = -\frac{S}{\nu}\mu + C$$

Because of symmetry considerations, in steady state, at the midplane $f(\mu)$ should be symmetric, and thus $\left.\frac{\partial f}{\partial \mu}\right|_{\mu=0} = 0 \Rightarrow C=0$

$$\frac{\partial f}{\partial \mu} = -\frac{S}{\nu}\frac{\mu}{1-\mu^2} \qquad f = \frac{S}{2\nu}\left[\ln(1-\mu^2) + C\right]$$

Since v was simply a parameter, we incorporate different BCs in the two different regimes.

$V < V_c$: $f=0$ at $\mu = \pm\mu_m \implies C = -\ln(1-\mu_m^2)$

$$f = \frac{S}{2\nu}\ln\left(\frac{1-\mu^2}{1-\mu_m^2}\right)$$

$V > V_c$: $f=0$ at $\mu = \pm\frac{V_{\parallel m}}{V} \implies C = -\ln\left(1-\frac{V_{\parallel m}^2}{V^2}\right)$

$$f = \frac{S}{2\nu}\ln\left(\frac{1-\mu^2}{1-\frac{V_{\parallel m}^2}{V^2}}\right)$$

$$f = \begin{cases} \frac{S(v)}{2\nu(v)}\ln\left(\frac{1-\mu^2}{1-\mu_m^2}\right) & V < V_c \\ \frac{S(v)}{2\nu(v)}\ln\left(\frac{1-\mu^2}{1-\frac{V_{\parallel m}^2}{V^2}}\right) & V > V_c \end{cases}$$

steady state solution f

Residence time: Define $\tau_r \approx \frac{f(t=0)}{\left|\frac{\partial f}{\partial t}(t=0)\right|}$ [when source=0] $f(t=0)$ already solved $= f$ in steady state

$\frac{\partial f}{\partial t}(t=0) = -S$ by definition, since we imagine turning off the source at $t=0$

Thus, $\tau_r \approx \frac{1}{2\nu}\left[\ln(1-\mu^2) - \ln(1-C(v))\right]$ $\qquad C(v) = \begin{cases} \mu_m^2 & V<V_c \\ \frac{V_{\parallel m}^2}{V^2} & V>V_c \end{cases}$

Consider $V<V_c$. $1-\mu_m^2 = 1-\left(1-\frac{1}{R}\right) = \frac{1}{R}$

$$\tau_r \approx \frac{1}{2\nu}\left[\ln(1-\mu^2) - \ln\left(\frac{1}{R}\right)\right] = \frac{1}{2\nu}\left[\ln(1-\mu^2) + \ln R\right]$$

And if we performed a suitable average of $\ln(1-\mu^2)$, it would be on the order of $\ln R$ (but negative), so for estimation just drop this term.

So $\tau_r \approx \frac{1}{2\nu}\ln R$