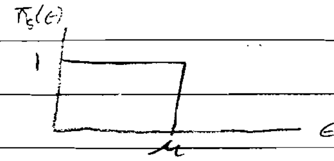
May 2003 #2 (SM)

a. degenerate electrons: Fermi-Dirac Distribution: $\bar{n}_s = \dfrac{1}{e^{\beta(\epsilon_s-\mu)}+1}$ # of particles in a state with a given energy ϵ

at $T \to 0$,



all states filled up to Fermi energy μ

corresponds to some Fermi momentum $p_F = \hbar k_F$, so $\mu = \dfrac{\hbar^2 k_F^2}{2m}$

volume of sphere in k-space: $\frac{4}{3}\pi k_F^3$

Density of states: $\rho(k)d^3k = \dfrac{V}{(2\pi)^3}$ # of translational states available between $\vec{k}$ and $\vec{k}+d\vec{k}$ (independent of spin)

$$2 \cdot \frac{V}{(2\pi)^3} \cdot \left(\frac{4}{3}\pi k_F^3\right) = N$$

$$\left[\text{explicitly, } 2\int_0^{\infty} f_{FD}(\epsilon)\rho(\epsilon)d\epsilon = N\right]$$

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

$$\mu = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

Total energy: $\sum_s n_s \epsilon_s \longrightarrow 2\int f_{FD}(\epsilon)\rho(\epsilon)\epsilon\, d\epsilon$

$$= 2\int_0^{\mu} \epsilon\rho(\epsilon)d\epsilon$$

$$\rho(\epsilon)d\epsilon = \rho(k(\epsilon))\left|\frac{dk}{d\epsilon}\right|d\epsilon$$

$$\rho(|\vec{k}|)dk = \frac{V}{(2\pi)^3} \cdot 4\pi k^2 dk = \frac{V}{2\pi^2}k^2 dk$$

$$k = \sqrt{\frac{2m}{\hbar^2}}\,\epsilon^{1/2} \qquad \frac{dk}{d\epsilon} = \sqrt{\frac{m}{2\hbar^2}}\,\epsilon^{-1/2}$$

$$\rho(\epsilon) = \frac{V}{2\pi^2} \cdot \frac{2m\epsilon}{\hbar^2} \cdot \sqrt{\frac{m}{2\hbar^2}}\,\epsilon^{-1/2} = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\,\epsilon^{1/2}$$

$$\bar{E} = \frac{2 \cdot V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^{\mu}\epsilon^{3/2}d\epsilon = \frac{\sqrt{2}\,V m^{3/2}}{\pi^2\hbar^3}\mu^{5/2}\frac{2}{5} = \frac{2\sqrt{2}\,V m^{3/2}}{5\pi^2\hbar^3}\,\frac{\hbar^5}{4\cdot\sqrt{2}\cdot m^{5/2}}\left(\frac{3\pi^2 N}{V}\right)^{5/3}$$

$$\text{total } E = \frac{3N\hbar^2\pi^2}{10 m_e}\left(\frac{3N}{\pi V}\right)^{2/3}$$

b. $U_{grav} = -\frac{3GM^2}{5R}$ binding energy for uniform density sphere

$U_K \sim \frac{1}{V^{2/3}} \sim \frac{1}{R^2}$

$E = \frac{\alpha}{R^2} - \frac{\beta}{R}$ $\quad \frac{dE}{dR} = \frac{-2\alpha}{R^3} + \frac{\beta}{R^2} = 0$ $\quad -2\alpha + \beta R = 0$

$\beta R = 2\alpha$

$R = \frac{2\alpha}{\beta}$

$E = \frac{\alpha\beta^2}{4\alpha^2} - \frac{\beta\beta}{2\alpha} = \frac{\beta^2}{4\alpha} - \frac{\beta^2}{2\alpha} = \boxed{\frac{-\beta^2}{4\alpha}}$ minimum energy

minimizing E: eqb $R = \frac{2\alpha}{\beta}$

$\alpha = \frac{3N(\hbar^2\pi^2)}{10m_c} \cdot \frac{3^{2/3}N^{2/3}}{(\pi\frac{4}{3}\pi)^{2/3}}$ $\quad \beta = \frac{3GM^2}{5}$

$M = 2Nm_p \quad N = \frac{M}{2m_p}$

$R_{eq} = \frac{\frac{2\cdot 3\pi^2\hbar^2}{10m_e}\left(\frac{9}{4\pi^2}\right)^{2/3}\frac{M^{5/3}}{(2m_p)^{5/2}}}{\frac{3GM^2}{5}} = \frac{\pi^2\hbar^2}{Gm_e(2m_p)^{5/2}}\left(\frac{9}{4\pi^2}\right)^{2/3} \cdot \frac{1}{M^{1/3}}$

$R \sim M^{-1/3}$

c. highly relativistic: $\epsilon = cp \Rightarrow$ same k_f

$\mu = c\hbar k_f = c\hbar\left(\frac{3\pi^2 N}{V}\right)^{1/3}$

total $E = 2\int_0^\mu \epsilon\,\rho(\epsilon)\,d\epsilon$ $\quad \rho(\epsilon)d\epsilon = \rho(k(\epsilon))\left|\frac{dk}{d\epsilon}\right|d\epsilon$

$\rho(k) = \frac{V}{2\pi^2}k^2$ $\quad k = \frac{\epsilon}{c\hbar}$ $\quad \frac{dk}{d\epsilon} = \frac{1}{c\hbar}$

$\rho(\epsilon) = \frac{V}{2\pi^2}\frac{\epsilon^2}{c^2\hbar^2}\cdot\frac{1}{c\hbar}$

$E = 2\cdot\frac{V}{2\pi^2}\frac{1}{c^3\hbar^3}\int_0^\mu \epsilon^3\,d\epsilon = \frac{V}{4\pi^2c^3\hbar^3}\mu^4 = \frac{Vc\hbar}{4\pi^2}(3\pi^2)^{4/3}\frac{N^{4/3}}{V^{4/3}}$

$E = \frac{3N\hbar\pi c}{4}\left(\frac{3N}{\pi V}\right)^{1/3}$

d. $E = \frac{3N\hbar\pi c}{4}\left(\frac{3N}{\pi\cdot\frac{4}{3}\pi}\right)^{1/3}\frac{1}{R}$

$U_g = -\frac{3GM^2}{5}\frac{1}{R}$

If $|U_G| > E$, then collapse occurs $\quad M = 2Nm_p \quad N = \frac{M}{2m_p}$

$$\frac{3\pi\hbar c}{4}\left(\frac{9}{4\pi^2}\right)^{1/3}\cdot\frac{M^{4/3}}{(2m_p)^{4/3}} < \frac{3GM^2}{5}$$

$$M^{\frac{2}{3}} > \frac{\hbar c}{G}\frac{5}{16}\frac{(9\pi)^{1/3}}{m_p^{4/3}}$$

$$M > \left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{5}{16}\right)^{3/2}\frac{3\pi^{1/2}}{m_p^2} = 3.4\cdot 10^{30}\ kg$$

$$M > 1.72\ M_\odot$$