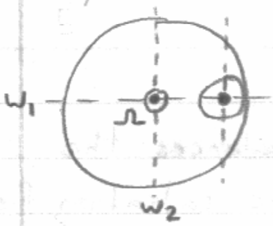
May 1999 CM #1



For the ball $I_{cm} = \frac{2}{5} m r^2 = I_1$
If we shift the reference point to the center of the cylinder:
$I_z = I_\theta = I_2 = I_1 + m(R-r)^2$ by the parallel axis theorm.

$$\vec{L} = I_z w_z \hat{z} + I_s w_s \hat{s} + I_\theta w_\theta \hat{\theta}$$
$$\vec{L} = I_2 \Omega \hat{z} + I_1 w_1 \hat{s} + I_2 w_2 \hat{\theta}$$
$$\frac{\partial \vec{L}}{\partial t} = I_2 \dot{\Omega} \hat{z} + I_1 \dot{w}_1 \hat{s} + I_2 \dot{w}_2 \hat{\theta} + I_1 w_1 \Omega \hat{\theta} - I_2 w_2 \Omega \hat{s}$$
$$\frac{\partial \vec{L}}{\partial t} = \vec{\tau} = \vec{r} \times (-mg\,\hat{z})$$
$$\vec{\tau} = ((R-r)\hat{s} + z\hat{z}) \times (-mg\,\hat{z})$$
$$\vec{\tau} = mg(R-r)\,\hat{\theta}$$

$\therefore$ the equations of motion are:

$\hat{z}$: $0 = I_2 \dot{\Omega}$ (1)

$\hat{s}$: $0 = I_1 \dot{w}_1 - I_2 w_2 \Omega$ (2)

$\hat{\theta}$: $mg(R-r) = I_2 \dot{w}_2 + I_1 w_1 \Omega$ (3)

a. from the first equation $\Omega = \text{const} = \Omega_0 = \frac{V}{R}$
Thus there is a rotating coordinate system with $\Omega = \frac{V}{R}$ where the ball moves only in the vertical direction.

b. (2) $\Rightarrow I_1 \ddot{w}_1 - I_2 \dot{w}_2 \Omega = 0 \Rightarrow I_2 \dot{w}_2 = \frac{1}{\Omega} I_1 \ddot{w}_1$ (4)

(3) $\Rightarrow I_2 \ddot{w}_2 + I_1 \dot{w}_1 \Omega = 0 \Rightarrow I_1 \dot{w}_1 = \frac{-1}{\Omega} I_2 \ddot{w}_2$ (5)

$$mg(R-r) = \frac{1}{\Omega} I_1 \ddot{w}_1 + I_1 w_1 \Omega$$
$$\frac{1}{I_1} \Omega mg(R-r) = \ddot{w}_1 + \Omega^2 w_1$$
$$\Rightarrow w_1 = C_1 \cos(\Omega t + \phi) + \frac{mg}{I_1 \Omega}(R-r)$$
$$w_1 = \frac{mg}{I_1 \Omega}(R-r)(1 - \cos(\Omega t)) \qquad w_1(t=0) = 0$$
$$0 = \frac{-1}{\Omega} I_2 \ddot{w}_2 - I_2 w_2 \Omega$$
$$0 = \ddot{w}_2 + \Omega^2 w_2$$
$$w_2 = \frac{mg}{I_1 \Omega}(R-r)\sin(\Omega t) \qquad (4) + w_2(t=0) = 0$$
$$v_z = w_2 r = \frac{-mg}{I_1 \Omega} r(R-r)\sin(\Omega t)$$
$$z(t) = \frac{mg}{I_1 \Omega^2} r(R-r)(\cos(\Omega t) - 1) \qquad z(0) = 0$$
$$\theta(t) = \Omega t = \frac{V}{R} t$$

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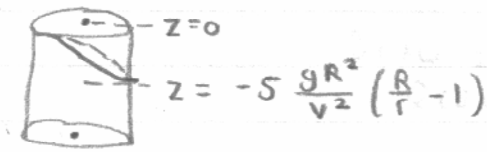
$$z(t) = \frac{mg}{mr^2} \cdot \frac{5}{2} \frac{R^2}{v^2} r(R-r)(\cos(\Omega t) - 1)$$

$$z(t) = \frac{5}{2} \frac{gR^2}{v^2}\left(\frac{R}{r} - 1\right)(\cos(\Omega t) - 1)$$



Ball oscillates back and forth between two values of z