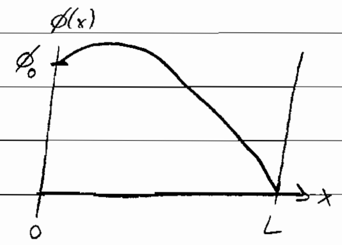
2004 Part II Q5 GPP I

a.



b. $\frac{d^2\phi}{dx^2} = -4\pi\rho = -4\pi q n$ (Poisson)

Continuity: $J = q n(x) v(x)$ $\quad q n(x) = \frac{J}{v(x)}$

Energy: $\frac{1}{2} m v(x)^2 + q\phi(x) = \frac{1}{2} m v_0^2 + q\phi$

$$v(x)^2 = v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))$$

$$\Rightarrow v(x) = \left[v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))\right]^{1/2}$$

$$\frac{d^2\phi}{dx^2} = \frac{-4\pi J}{v(x)} = \frac{-4\pi J}{\left[v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))\right]^{1/2}}$$

multiply by $\frac{d\phi}{dx}$:

$$\frac{1}{2}\frac{d}{dx}\left[\left(\frac{d\phi}{dx}\right)^2\right] = \frac{-4\pi J \frac{d\phi}{dx}}{\left[v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))\right]^{1/2}}$$

integrate in x:

$$\left(\frac{d\phi}{dx}\right)^2 = -8\pi J \cdot 2 \cdot \left(\frac{-m}{2q}\right)\left[v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))\right]^{1/2} + C$$

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{8\pi J m}{q}\underbrace{\left[v_0^2 + \frac{2q}{m}(\phi(0) - \phi(x))\right]^{1/2}}_{v(x)} + C$$

$\Rightarrow$

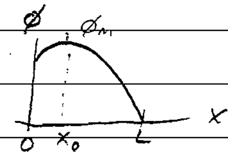
$$\alpha = \frac{8\pi J m}{q} \cdot \left(\frac{2q}{m}\right)^{1/2} = 8\pi J\left(\frac{2m}{q}\right)^{1/2} = 16\pi J\left(\frac{m}{2q}\right)^{1/2}$$

$$\phi_M = \phi(0) + \frac{m v_0^2}{2q}$$

c. where $\frac{d\phi}{dx} = 0$ (0 electric field), maximum steady state is reached with $v(x) = 0$. Otherwise additional ions could be injected **until** the ions are slowed down to 0; a limit is reached when any additional ions would cause repulsion back to $x=0$.

since $\left(\frac{d\phi}{dx}\right)^2 = \frac{8\pi J m}{q} v(x) + C$, $\quad C = 0$ $(\beta = 0)$

d. $\left(\frac{d\phi}{dx}\right)^2 = \alpha\left[\phi_m - \phi\right]^{1/2}$



There is some x_0 where ϕ reaches a maximum. We have argued that at this point, $V=0$. Then by conservation of energy, we have

$$\frac{mv_0^2}{2q} = -\left(\phi(0) - \phi(x_0)\right)$$

$\Rightarrow$ For $0 < x < x_0$, $\quad \frac{d\phi}{dx} = \alpha^{1/2}\left[\phi_m - \phi\right]^{1/4}$

For $x_0 < x < L$, $\quad \frac{d\phi}{dx} = -\alpha^{1/2}\left[\phi_m - \phi\right]^{1/4}$

$0<x<x_0$: $\frac{d\phi}{[\phi_m - \phi]^{1/4}} = \alpha^{1/2}dx \longrightarrow -\frac{4}{3}\left[\phi_m - \phi(x)\right]^{3/4} = \alpha^{1/2}x + C_1$

at $x=0$: $-\frac{4}{3}\left[\frac{mv_0^2}{2q} + \phi(0) - \phi(0)\right]^{3/4} = C_1 \qquad C_1 = -\frac{4}{3}\left[\frac{mv_0^2}{2q}\right]^{3/4}$

at $x=x_0$: $\quad 0 = \alpha^{1/2}x_0 + C_1 \qquad (\phi_m - \phi = 0 \text{ at } x = x_0)$

$$\Longrightarrow \alpha^{1/2}x_0 = \frac{4}{3}\left[\frac{mv_0^2}{2q}\right]^{3/4}$$

$x_0 < x < L$: $\frac{-d\phi}{[\phi_m - \phi]^{1/4}} = \alpha^{1/2}dx \Longrightarrow \frac{4}{3}\left[\phi_m - \phi(x)\right]^{3/4} = \alpha^{1/2}x + C_2$

at $x=x_0$: $\quad 0 = \alpha^{1/2}x_0 + C_2 \qquad$ substitute $\alpha^{1/2}x_0$ from above

$$C_2 = -\frac{4}{3}\left[\frac{mv_0^2}{2q}\right]^{3/4}$$

at $x=L$: $\frac{4}{3}\left[\frac{mv_0^2}{2q} + \underbrace{\phi(0) - \phi(L)}_{\phi_0}\right]^{3/4} = \alpha^{1/2}L - \frac{4}{3}\left[\frac{mv_0^2}{2q}\right]^{3/4}$

$\alpha^{1/2} = \frac{4}{3L}\left\{\left(\frac{mv_0^2}{2q}\right)^{3/4} + \left(\frac{mv_0^2}{2q} + \phi_0\right)^{3/4}\right\} \qquad \alpha = 16\pi J\left(\frac{m}{2q}\right)^{1/2}$

$$16\pi J\left(\frac{m}{2q}\right)^{1/2} = \frac{16}{9L^2}\left[\left(\frac{mv_0^2}{2q}\right)^{3/4} + \left(\frac{mv_0^2}{2q} + \phi_0\right)^{3/4}\right]^2$$

$$\boxed{J = \frac{\left(\frac{2q}{m}\right)^{1/2}}{9\pi L^2}\left[\left(\frac{mv_0^2}{2q}\right)^{3/4} + \left(\frac{mv_0^2}{2q} + \phi_0\right)^{3/4}\right]^2}$$

For $V_0 \to 0$, we obtain $J = \frac{\left(\frac{2q}{m}\right)^{1/2}}{9\pi L^2}\phi_0^{3/2}$ <u>the $\frac{3}{2}$ scaling law</u>