1999 Part II Q5 GPP I

$$\frac{\partial f}{\partial t} = \Gamma\left[\frac{1}{v^2}\frac{\partial}{\partial v} f + \frac{1+Z_i}{2v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} f\right] + S(v,\mu,t)$$

v = electron speed $\mu = \frac{v_z}{v}$ S = source

$$S = (1-\mu^2)\dot{g}(v)$$

a. • Slowing down term + pitch angle scattering term

b. conserve particles? $\int_0^\infty dv\, v^2 \int_{-1}^{1} d\mu$ is the volume element (up to 2π)

$\frac{1}{v^2}\frac{\partial f}{\partial v}$: $\int d\mu \int_0^\infty dv\, v^2 \frac{1}{v^2}\frac{\partial f}{\partial v} = \int d\mu\, f\Big|_{v=0}^{v=\infty} = -\int d\mu\, f(v=0,\mu) = 0$ only if $f(v=0)=0$

$\frac{1}{v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}$ integrate by μ get 0 f

Does not conserve particles because in general $f(v=0) \neq 0$

conserve entropy? entropy $\propto -f\ln f$ $\frac{\partial}{\partial t}(f\ln f) = (1+\ln f)\dot{f}$ $\dot{f}$ term does not conserve particles, but let's check $(\ln f)\dot{f}$ anyway

Apply $\int_0^\infty dv\, v^2 \int_{-1}^{1} d\mu \ln f [\,]$ $\frac{1}{v^2}\frac{\partial f}{\partial v}$: $\int d\mu \int_0^\infty \ln f \frac{\partial f}{\partial v} dv \rightarrow \int_0^\infty \left(\frac{\partial}{\partial v}[f\ln f] - f\frac{\partial}{\partial v}\ln f\right) dv$

$$\rightarrow f\ln f\Big|_{v=0}^{v=\infty} - f\Big|_{v=0}^{v=\infty} = 0$$

$\frac{1}{v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} f$: $\int_0^\infty \frac{dv}{v}\int_{-1}^{1} d\mu \ln f \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} \rightarrow \int \frac{dv}{v}\int_{-1}^{1} d\mu \left(\underbrace{\frac{\partial}{\partial \mu}\left[\ln f (1-\mu^2)\frac{\partial f}{\partial \mu}\right]}_{0} - (1-\mu^2)\frac{\partial f}{\partial \mu}\frac{\partial}{\partial \mu}\ln f\right)$

$$\rightarrow -\int \frac{dv}{v}\int_{-1}^{1} d\mu\, \underbrace{(1-\mu^2)\frac{1}{f}\left(\frac{\partial f}{\partial \mu}\right)^2}_{>0}$$

Does not conserve entropy

conserve energy? $\int dv\, v^2 \int_{-1}^{1} d\mu\, v^2$

$\frac{1}{v^2}\frac{\partial f}{\partial v}$: $\int d\mu \int dv\, v^2 \frac{\partial f}{\partial v} = \int dv\left(\frac{\partial}{\partial v}(v^2 f) - 2fv\right) \Rightarrow \underbrace{-2\int d\mu \int dv\, fv}_{<0}$ energy decreases

$\frac{1}{v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}$: $\int_0^\infty dv\, v \int_{-1}^{1} d\mu \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} = 0$

Does not conserve energy

c. Steady state: $\frac{\partial f}{\partial t} = 0$

$$\Gamma\left[\frac{1}{v^2}\frac{\partial f}{\partial v} + \frac{1+Z_i}{2v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}\right] = -(1-\mu^2)\dot{g}(v)$$

steady state distribution of electron speeds: $f_v(v) = \int_{-1}^{1} d\mu\, f(v,\mu)$

Apply $\int_{-1}^{1} d\mu\,[\ \ ]$

$$\Rightarrow \Gamma\left[\frac{1}{v^2}\frac{\partial f_v}{\partial v}\right] = -\dot{g}(v)\int_{-1}^{1}(1-\mu^2)\,d\mu = -\dot{g}(v)\cdot\frac{4}{3}$$

$$\frac{\partial f_v}{\partial v} = -v^2\dot{g}(v)\cdot\frac{4}{3\Gamma}$$

$$f_v(v) = -\frac{4}{3\Gamma}\int dv\, v^2 g(v) + C$$

where C is determined by boundary conditions: $f_v(v\to\infty) = 0$

Integrate the diff. eqn. from $v' = v$ to $v = \infty$

$$\underbrace{\cancel{f_v(\infty)}}_{0} - f_v(v) = -\frac{4}{3\Gamma}\int_v^\infty v'^2\dot{g}(v')\,dv'$$

$$\Rightarrow \boxed{f_v(v) = \frac{4}{3\Gamma}\int_v^\infty v'^2\dot{g}(v')\,dv'}$$

d. Since the $P_\ell(\mu)$ are eigenfunctions of the operator $\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial P_\ell}{\partial\mu} = -\ell(\ell+1)P_\ell(\mu)$, we expand f in $P_\ell(\mu)$. $f = \sum_\ell f_\ell(v)P_\ell(\mu)$. Since the inhomogeneous term has only P_0, P_2, we need only keep those terms

$$f = f_0(v)P_0(\mu) + f_2(v)P_2(\mu)$$

$$\Rightarrow \frac{1}{v^2}P_0\frac{\partial f_0}{\partial v} + \frac{1}{v^2}P_2\frac{\partial f_2}{\partial v} + \cancel{\frac{1+Z_i}{2v^3}f_0\cdot(-0)(0+1)P_0} + \frac{1+Z_i}{2v^3}f_2(-2)(2+1)P_2$$
$$= -\frac{2}{3}(P_0-P_2)\frac{\dot{g}(v)}{\Gamma}$$

And the P_ℓ are orthogonal, so match terms:

$$\frac{1}{v^2}\frac{df_0}{dv} = -\frac{2}{3}\frac{\dot{g}(v)}{\Gamma} \qquad \frac{df_0}{dv} = -\frac{2}{3\Gamma}\dot{g}(v)\,v^2 \qquad f_0(v\to\infty)\to 0$$

Integrate from v to ∞: $\quad f_0(v) = \frac{2}{3\Gamma}\int_v^\infty dv'\,\dot{g}(v')\,v'^2$

$$\frac{1}{v^2}\frac{df_2}{dv} - \frac{3(1+Z_i)}{v^3}f_2 = \frac{2}{3}\frac{\dot{g}}{\Gamma}$$

$$\frac{df_2}{dv} - \frac{3(1+Z_i)}{v}f_2 = \frac{2}{3}\frac{\dot{g}}{\Gamma}v^2$$

multiply by integrating factor $e^{-3(1+Z_i)\int\frac{dv}{v}} = v^{-3-3Z_i}$

$$\frac{d}{dv}\left[v^{-3-3Z_i}f_2\right] = \frac{2}{3}\frac{\dot{g}}{\Gamma}\frac{1}{v^{1+3Z_i}}$$

integrate from $v'=v$ to $v'=\infty$

$f_2(v\to\infty)=0$

$$\frac{-f_2}{v^{3+3Z_i}} = \frac{2}{3\Gamma}\int_v^\infty dv'\,\frac{\dot{g}}{v'^{1+3Z_i}}$$

$$f_2 = \frac{-2}{3\Gamma}v^{3+3Z_i}\int_v^\infty dv'\,\frac{\dot{g}}{v'^{1+3Z_i}}$$

Deviation from isotropy in steady state:

$f_2(v)P_2(\mu) \qquad f_2(v) = -\frac{2}{3\Gamma}v^{3+3Z_i}\int_v^\infty dv'\,\frac{\dot{g}(v')}{v'^{1+3Z_i}}$

largest deviation (f_2 largest) at higher speeds

e. $\dot{g}(v) = k\,\delta(v-v_0) \qquad f_0(v) = \frac{2k}{3\Gamma}\int_v^\infty dv'\,v'^2\,\delta(v'-v_0) = \begin{cases}\frac{2kv_0^2}{3\Gamma} & v<v_0\\ 0 & v>v_0\end{cases}$

$$f_2(v) = -\frac{2k}{3\Gamma}v^{3+3Z_i}\int_v^\infty dv'\,\frac{\delta(v'-v_0)}{v'^{1+3Z_i}}$$

$$f_2(v) = \begin{cases}\frac{-2k}{3\Gamma}\frac{v^{3(1+Z_i)}}{v_0^{1+3Z_i}} & v<v_0\\ 0 & v>v_0\end{cases}$$

$$f(v,\mu) = \begin{cases}\frac{2kv_0^2}{3\Gamma}\left[1-\frac{(3\mu^2-1)}{2}\left(\frac{v}{v_0}\right)^{3(1+Z_i)}\right] & v<v_0\\ 0 & v>v_0\end{cases}$$