Seth Dorfman   Prelims   December 30, 2005

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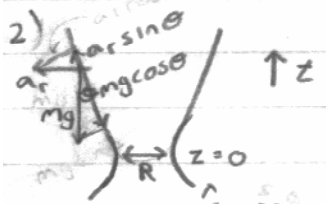

$x^2+y^2=r^2=R^2+z^2$

$\frac{d^2}{dt^2}(r\hat{r})=\frac{d}{dt}(\dot{r}\hat{r}+r\dot{\phi}\hat{\phi})$

$\vec{a}=\ddot{r}\hat{r}+2\dot{r}\dot{\phi}\hat{\phi}+r\ddot{\phi}\hat{\phi}-r\dot{\phi}^2\hat{r}$

$\vec{a}=(\ddot{r}-r\dot{\phi}^2)\hat{r}+\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\phi}$

$\vec{r}(t)=r_0+\varepsilon\sin(\omega t)$

No forces in $\hat{\phi}$ direction $\Rightarrow \frac{d}{dt}(r^2\dot{\phi})=0$

$\Rightarrow \frac{d}{dt}(\dot{\phi}(r_0^2+2r_0\varepsilon\sin(\omega t)+\underbrace{\varepsilon^2\sin^2(\omega t)}_{small}))=0$

$\dot{\phi}=\Omega(1-\frac{2\varepsilon}{r_0}\sin(\omega t))$

$\ddot{r}=-\varepsilon\omega^2\sin(\omega t)$

$r\dot{\phi}^2=\Omega^2(1-\frac{2\varepsilon}{r_0}\sin(\omega t))^2(r_0+\varepsilon\sin(\omega t))$

$=\Omega^2(r_0-\frac{4\varepsilon}{\cancel{r_0}}\sin(\omega t)\cdot\cancel{r_0}+\varepsilon\sin(\omega t))$

$r\dot{\phi}^2=\Omega^2(r_0-3\varepsilon\sin(\omega t))$

Force balance along the surface:

$ma_r\sin\theta=-mg\cos\theta$ $\tan\theta=\frac{dr}{dz}$ $2r\frac{dr}{dz}=2z$

$ma_r\tan\theta=-mg$ $\tan\theta=\sqrt{1-(\frac{R}{r})^2}$ $\frac{dr}{dz}=\frac{z}{r}=\frac{\sqrt{r^2-R^2}}{r}$

$\cancel{m}(\ddot{r}-r\dot{\phi}^2)\sqrt{1-(\frac{R}{r})^2}=-\cancel{m}g$

$\sqrt{1-(\frac{R}{r})^2}=\sqrt{1-(\frac{R}{r_0+\varepsilon\sin(\omega t)})^2}$

$=\sqrt{1-(\frac{R}{r_0})^2(\frac{1}{1+\varepsilon\sin(\omega t)/r_0})^2}$

$\approx\sqrt{1-(\frac{R}{r_0})^2(1-2\frac{\varepsilon}{r_0}\sin(\omega t))}$

$=\sqrt{1-(\frac{R}{r_0})^2+2(\frac{R}{r_0})^2\frac{\varepsilon}{r_0}\sin(\omega t)}$

$=\sqrt{1-(\frac{R}{r_0})^2}\left[1+2(\frac{R}{r_0})^2\frac{\varepsilon}{r_0}\sin(\omega t)/(1-(\frac{R}{r_0})^2)\right]^{1/2}$

$\sqrt{1-(\frac{R}{r})^2}\approx\sqrt{1-(\frac{R}{r_0})^2}\left[1+(\frac{R}{r_0})^2\frac{\varepsilon}{r_0}\sin(\omega t)/(1-(\frac{R}{r_0})^2)\right]$

$\therefore(-\varepsilon\omega^2\sin(\omega t)-\Omega^2(r_0-3\varepsilon\sin(\omega t)))\sqrt{1-(\frac{R}{r_0})^2}$

$=-\cancel{m}g\left[1-(\frac{R}{r_0})^2\frac{\varepsilon}{r_0}\sin(\omega t)/(1-(\frac{R}{r_0})^2)\right]$

Subtracting the 0^{th} order equation ($\cancel{m}\Omega^2r_0\sqrt{1-(\frac{R}{r_0})^2}=mg$)

and canceling $\varepsilon\sin(\omega t)$:

$(3\Omega^2-\omega^2)\sqrt{1-(\frac{R}{r_0})^2}=\cancel{m}g(\frac{R}{r_0})^2\frac{1}{r_0}/(1-(\frac{R}{r_0})^2)$

$3\Omega^2-\omega^2=\frac{\cancel{m}gR^2}{(r_0^2-R^2)^{3/2}}$

(over)

$$w^2 = 3\Omega^2 - \frac{gR^2}{(r_0^2 - R^2)^{3/2}} \qquad \Omega^2 = \frac{g}{(r_0^2 - R^2)^{1/2}}$$

$$w^2 = \frac{3g}{(r_0^2 - R^2)^{1/2}} - \frac{gR^2}{(r_0^2 - R^2)^{3/2}}$$

$$w^2 = \frac{g}{\sqrt{r_0^2 - R^2}}\left(3 - \frac{R^2}{r_0^2 - R^2}\right)$$

$$w^2 = \frac{g}{\sqrt{r_0^2 - R^2}}\left(\frac{3r_0^2 - 3R^2 - R^2}{r_0^2 - R^2}\right)$$

$$w^2 = \frac{g(3r_0^2 - 4R^2)}{(r_0^2 - R^2)^{3/2}}$$

The orbit is unstable if $3r_0^2 < 4R^2$

$$r_0^2 < \frac{4}{3}R^2 \qquad (r^2 = R^2 + z^2)$$

$$R^2 + z^2 < \frac{4}{3}R^2$$

$$z^2 < \frac{1}{3}R^2$$

$\therefore$ unstable orbit $\Rightarrow |z| < \frac{R}{\sqrt{3}}$