2012, Part 1, Question 3 (jburby@princeton.edu)

(a) When $k > 0$, the Landau contour is the directed curve in the complex u-plane used to analytically continue the dielectric function $\epsilon(\omega, k)$ from the upper-half ω-plane to the lower half. When $k < 0$, it serves to continue ϵ from the lower-half to the upper-half. The shape is determined by demanding that the pole in $1/(w/k - u)$ cannot cross the contour as the imaginary part of ω is dragged through 0.

A way to remember this is to first recall that when using Laplace transforms to solve the initial value formulation of the problem, the imaginary part of ω is assumed to be sufficiently positive to ensure convergence of the Laplace transform. Thus, ω/k begins with an imaginary part whose sign is equal to the sign of k.

(b) For stability, the imaginary part of ω that solves the dispersion relation must be less that or equal to 0. So assume the contrary, i.e. $\text{Im}(\omega) > 0$. Then the Landau contour is along the real u-axis. This means the dispersion relation becomes

$$\begin{aligned}
0 =& 1 + \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{g'(u)}{\frac{\omega}{k} - u} du \\
=& 1 + \frac{k}{\omega}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{(\frac{\omega}{k} - u + u)g'(u)}{\frac{\omega}{k} - u} du \\
=& 1 + \frac{k}{\omega}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} g'(u)du + \frac{k}{\omega}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{ug'(u)}{\frac{\omega}{k} - u} du \\
=& 1 + \frac{k}{\omega}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{ug'(u)(\frac{\omega^*}{k} - u)}{\left|\frac{\omega}{k} - u\right|^2} du.
\end{aligned}$$

Multiplying the previous expression by ω and then taking the imaginary part finally gives

$$0 = \text{Im}(\omega)\left(1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{ug'(u)}{\left|\frac{\omega}{k} - u\right|^2} du\right).$$

Because $ug'(u) < 0$, the second factor in this expression must be non-zero. This implies $\text{Im}(\omega) = 0$, contradicting our assumption that $\text{Im}(\omega) > 0$ (remember that this was what ensured the Landau contour was along the real u-axis). So it can only be the case that $\text{Im}(\omega) \leq 0$, implying stability.