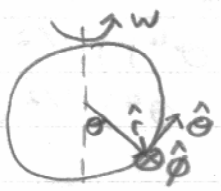
Seth Dorfman Prelims December 21, 2005

May 2000 CM

3)

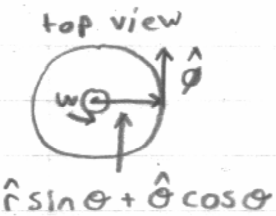


$\vec{r} \equiv$ vector from center of hoop to bead $|\vec{r}| = a$

$\frac{\partial \vec{r}}{\partial t}\Big|_{\text{lab frame}} = \frac{\partial \vec{r}}{\partial t}\Big|_{\text{rotating frame}} + \vec{w} \times \vec{r}$

$\vec{V} = \frac{\partial}{\partial t}(a\hat{r}) + \vec{w} \times (a\hat{r})$

$\vec{V} = a\dot{\theta}\hat{\theta} + aw\sin\theta\hat{\phi}$



$\frac{\partial \vec{V}}{\partial t}\Big|_{lab} = \frac{\partial \vec{V}}{\partial t}\Big|_{rot} + \vec{w} \times \vec{V}$

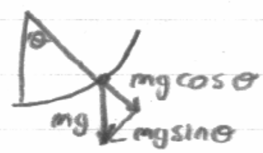
Note: $\hat{\phi}$ is fixed in the rotating frame.

$\vec{a} = \frac{\partial}{\partial t}(a\dot{\theta}\hat{\theta}) + \vec{w} \times (a\dot{\theta}\hat{\theta} + aw\sin\theta\hat{\phi})$

$\vec{a} = a\ddot{\theta}\hat{\theta} - a\dot{\theta}^2\hat{r} + aw\dot{\theta}\cos\theta\hat{\phi} - aw^2\sin\theta(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$

$\vec{a} = \hat{r}(-a\dot{\theta}^2 - aw^2\sin^2\theta) + \hat{\theta}(a\ddot{\theta} - aw^2\sin\theta\cos\theta)$
$+ \hat{\phi}(aw\dot{\theta}\cos\theta)$

In the $\hat{\theta}$ direction:



$ma_\theta = \cancel{m}\cancel{m}g\sin\theta$

$a\ddot{\theta} - aw^2\sin\theta\cos\theta = -g\sin\theta$

$(\ddot{\theta} - w^2\sin\theta\cos\theta + \frac{g}{a}\sin\theta = 0) \times \dot{\theta}$

$\frac{d}{dt}(\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}w^2\sin^2\theta - \frac{g}{a}\cos\theta) = 0$

$\frac{E}{ma^2} = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}w^2\sin^2\theta - \frac{g}{a}\cos\theta = \text{const}$

$E = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2w^2\sin^2\theta - mga\cos\theta$

$V_{eff} = -\frac{1}{2}ma^2w^2\sin^2\theta - mga\cos\theta$

$w^2 = \frac{g}{a} \Rightarrow V_{eff} = -mga(\frac{1}{2}\sin^2\theta + \cos\theta)$

For small θ: $\sin\theta = \theta - \frac{\theta^3}{6}$

$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$

$V_{eff} = -mga(\frac{1}{\cancel{2}}\theta^2 - \frac{\theta^4}{6} + 1 - \frac{\theta^2}{\cancel{2}} + \frac{\theta^4}{24})$

$= -mga(1 - \theta^4(\frac{4-1}{24}))$

$V_{eff} = -mga(1 - \frac{\theta^4}{8})$

Let θ_0 be the amplitude of the small oscillations

Then $E = -mga(1 - \frac{\theta_0^4}{8})$

$\frac{1}{2}ma^2\dot{\theta}^2 = E - V_{eff}$

$\dot{\theta} = \sqrt{\frac{2}{ma^2}(E - V_{eff})} = \frac{d\theta}{dt}$

$$\frac{dt}{d\theta} = a\sqrt{\frac{m}{2}}\,\frac{1}{\sqrt{E - V_{eff}}}$$

period $T = \int_{-\theta_0}^{\theta_0} a\sqrt{\frac{m}{2}}\,\frac{1}{\sqrt{E - V_{eff}}}\,d\theta$

$$= a\sqrt{\frac{m}{2}}\int_{-\theta_0}^{\theta_0}\left(\frac{mga}{8}(\theta_0^4 - \theta^4)\right)^{-1/2} d\theta$$

$$T = a\sqrt{\frac{m}{2}}\cdot\sqrt{\frac{8}{mga}}\int_{-\theta_0}^{\theta_0}(\theta_0^4 - \theta^4)^{-1/2}\,d\theta$$

Let $x = \frac{\theta}{\theta_0}$ $\quad dx = \frac{1}{\theta_0}\,d\theta$

$$T = 2\sqrt{\frac{a}{g}}\int_{-1}^{1}(\theta_0^4 - (x\theta_0)^4)^{-1/2}\,\theta_0\,dx$$

$$= 2\sqrt{\frac{a}{g}}(\theta_0^4)^{-1/2}\,\theta_0\int_{-1}^{1}\frac{1}{\sqrt{1-x^4}}\,dx$$

$$T = \frac{2}{\theta_0}\sqrt{\frac{a}{g}}\int_{-1}^{1}\frac{1}{\sqrt{1-x^4}}\,dx$$