2007 Part I Q6a

$$\frac{d^2y}{dz^2} + z^2 y = 1$$

a. Asymptotic behaviors for $z \to 0$

First, homogeneous solutions: $\frac{d^2y}{dz^2} + z^2 y = 0$

Can use $y = e^S$: $y' = S'e^S$ $y'' = [S'' + (S')^2]e^S$

$$S'' + (S')^2 + z^2 = 0$$

dominant balance: $S'' + (S')^2 = 0$ $\frac{(S')'}{(S')^2} = -1$ $\frac{-1}{S'} = -z + c_1$

$$S' = \frac{1}{z + C_1} \qquad S = \ln(z + c_1) + C_2$$

$y = e^S = Az + B$ homogeneous solution

Particular solution: $\frac{d^2y}{dz^2} = 1$ dominant balance $y_p = \frac{1}{2}z^2$

$$y = \frac{1}{2}z^2 + Az + B$$

b. For $z \to \infty$

(i) particular solution: $y = \frac{1}{z^2}$

(ii) homogeneous solution: $\frac{d^2y}{dz^2} + z^2 y = 0$ $S'' + (S')^2 + z^2 = 0$

dominant: $(S')^2 + z^2 = 0$ $S' = \pm iz$

$S' = \pm iz + g'$ $S'' = \pm i + g''$

$(S')^2 = -z^2 \pm 2izg' + (g')^2$

$$\pm i + g'' - \cancel{z^2} \pm 2izg' + (g')^2 + \cancel{z^2} = 0$$

$\pm 1 \pm 2zg' = 0$ $g' = -\frac{1}{2z}$

$s' = \pm iz - \frac{1}{2z}$

$s = \pm i\frac{1}{2}z^2 - \frac{1}{2}\ln z$

$y = e^S = \frac{1}{\sqrt{z}} e^{\pm i \frac{1}{2} z^2}$

$$y = \frac{1}{\sqrt{z}}\cos\left(\frac{1}{2}z^2\right), \quad \frac{1}{\sqrt{z}}\sin\left(\frac{1}{2}z^2\right)$$

c. Integral representation. Try $y = \int_C e^{z^2 t} f(t)\,dt$

$y' = \int 2zt\, e^{z^2 t} f(t)\,dt$ $\qquad$ $y'' = \int (2t + 4z^2t^2) e^{z^2 t} f(t)\,dt$

$\int (2t + 4z^2t^2 + z^2) e^{z^2 t} f(t)\,dt = 1$

$$z^2 e^{z^2 t} = \frac{d}{dt} e^{z^2 t}$$

$\Rightarrow \int \left[f(2t) + f(4t^2+1)\frac{d}{dt} \right] e^{z^2 t}\,dt = 1$

int by parts: $f(4t^2+1)\frac{d}{dt} e^{z^2 t} = f(4t^2+1)e^{z^2 t} - \left[f'(4t^2+1) + f(8t) \right] e^{z^2 t}$

$\Rightarrow f(4t^2+1) e^{z^2 t} \Big|_a^b = 1$ $\qquad$ $\int_C \left[(2t - 8t) f - f'(4t^2+1) \right] e^{z^2 t}\,dt = 0$

$$-6tf - f'(4t^2+1) = 0$$

$f'(4t^2+1) = -6tf$ $\qquad$ $\frac{f'}{f} = \frac{-6t}{4t^2+1}$

$\ln f = -\frac{6}{8} \ln(4t^2+1) + c$

$f = C(4t^2+1)^{-3/4}$

endpoint contribution: $C(4t^2+1)^{1/4} e^{z^2 t} \Big|_a^b = 1$

Choose one of the endpoints, $b = 0 \Rightarrow$ get $C = 1$

choose endpoint a: $t = -\infty$ (for z=real), get 0

For z complex can use $a = -\frac{\infty}{z^2}$, or $\pm \frac{i}{2}, \ldots$

$$y = \int_{-\infty}^{0} dt\, \frac{e^{z^2 t}}{(4t^2+1)^{3/4}}$$