Jan 1999 #1 (SM)

Note: part (a) can be done in a much simpler way by using the grand partition function

Weirdon: state may contain 0, 1, or 2 particles

1D, line of length L, reservoir τ, mass m

a. occupancy for a state of energy ϵ, when the chemical potential is μ

(Analog of FD calculation)

Avg. value of n_s: $\bar{n}_s = \dfrac{\sum_{\text{all states}} n_s \cdot P(n_s)}{\sum P(n_s)} = \dfrac{\sum_R n_s e^{-\beta E_R}}{\sum_R e^{-\beta E_R}}$ $\qquad \beta = \frac{1}{k\tau}$

R = all possible states

$$\bar{n}_s = \frac{\sum_R n_s e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \cdots)}}{\sum_R e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \cdots)}}$$ subject to $\sum_r n_r = N$ (fixed number of weirdons)

$$= \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum_{n_{i\neq s}}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum_{n_{i\neq s}}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}$$

For the second sum, pick a value for n_s, then perform the sum

when there are M particles remaining for states not equal to s, call the sum

$$\sum_{n_{i\neq s}}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)} \equiv Z_s(M)$$

$$\Rightarrow \bar{n}_s = \frac{0 + e^{-\beta\epsilon_s} Z_s(N-1) + 2e^{-2\beta\epsilon_s} Z_s(N-2)}{Z_s(N) + e^{-\beta\epsilon_s} Z_s(N-1) + e^{-2\beta\epsilon_s} Z_s(N-2)}$$

For $\Delta N \ll N$, $\ln Z_s(N-\Delta N) = \ln Z_s(N) - \frac{\partial \ln Z_s}{\partial N}\Delta N = \ln Z_s(N) - \alpha_s \Delta N$

$\Rightarrow Z_s(N-\Delta N) = Z_s(N) e^{-\alpha_s \Delta N}$

where $\alpha_s \equiv \dfrac{\partial \ln Z_s}{\partial N}$

$Z_s(N)$ is a sum over many states, so its logarithm should not depend much on which state s is removed

$\Rightarrow$ Approximate $\alpha_s \to \alpha$ independent of s

$\Rightarrow \alpha = \dfrac{\partial \ln Z}{\partial N}$ $\qquad Z$: sum over all states

$$\Rightarrow \bar{n}_s = \frac{Z_s(N)\left[e^{-\beta\epsilon_s - \alpha} + 2e^{-2\beta\epsilon_s - 2\alpha}\right]}{Z_s(N)\left[1 + e^{-\beta\epsilon_s - \alpha} + e^{-2\beta\epsilon_s - 2\alpha}\right]}$$

let $\alpha = -\beta\mu$

$$\bar{n}_s = \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}$$ # of particles per state

for $\frac{\mu-\epsilon}{\tau} = -\infty$ $(\epsilon > \mu,\ \tau = 0)$ $\bar{n}_s = 0$

$\frac{\mu-\epsilon}{\tau} = 0$ $\epsilon = \mu$ $\bar{n}_s = \frac{1+2}{1+1+1} = 1$

$\frac{\mu-\epsilon}{\tau} = \infty$ $\epsilon < \mu,\ \tau = 0$ $\bar{n}_s = \frac{2e^{2\beta(\mu-\epsilon)}}{e^{2\beta(\mu-\epsilon)}} = 2$

b. density of states (# of states per unit energy, as a function of energy)

assume spin 0

$\rho(k)dk = \frac{L}{2\pi}$ # of states between k and $k+dk$ $-\infty < k < \infty$

$\rho(|k|)d|k| = \frac{L}{\pi}$ $0 < |k| < \infty$

$\rho(\epsilon)d\epsilon = \rho(|k(\epsilon)|)\frac{dk}{d\epsilon}d\epsilon$ # of states between ϵ, $\epsilon + d\epsilon$

$\epsilon = \frac{\hbar^2 k^2}{2m}$ $|k| = \sqrt{\frac{2m}{\hbar^2}}\,\epsilon^{1/2}$ $\frac{dk}{d\epsilon} = \sqrt{\frac{m}{2\hbar^2}}\,\epsilon^{-1/2}$

$\rho(\epsilon)d\epsilon = \frac{L}{\pi}\cdot\sqrt{\frac{m}{2\hbar^2}}\cdot\epsilon^{-1/2}\,d\epsilon$

$\rho(\epsilon) = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2}}\,\epsilon^{-1/2}$ # of states per unit energy

c. Suppose weirdon gas is cold $(\tau \to 0)$, and contains N weirdons. Calculate μ.

$N = \int_0^\infty d\epsilon\ \underbrace{n_s(\epsilon)\rho(\epsilon)}_{\text{\# of particles per unit energy}}$ $n_s(\epsilon) = \begin{matrix} 2 & \epsilon < \mu \\ 0 & \epsilon > \mu \end{matrix}$ at $\tau = 0$

$N = \int_0^\mu d\epsilon\, 2\cdot\frac{L}{\pi\hbar}\sqrt{\frac{m}{2}}\,\epsilon^{-1/2} = \frac{L}{\pi\hbar}\sqrt{2m}\cdot 2\,\epsilon^{1/2}\Big|_0^\mu = \sqrt{8m}\,\frac{L}{\pi\hbar}\,\mu^{1/2}$

$\mu = \frac{N^2\pi^2\hbar^2}{8mL^2}$ (or Fermi "sphere" calculation)

d. $E = \int_0^\infty d\epsilon\ n_s(\epsilon)\rho(\epsilon)\cdot\epsilon = \frac{L}{\pi\hbar}\sqrt{2m}\int_0^\mu d\epsilon\ \epsilon^{1/2} = \frac{L}{\pi\hbar}\sqrt{2m}\,\frac{2}{3}\,\epsilon^{3/2}\Big|_0^\mu$

$$E = \frac{2\sqrt{2m}}{3} \frac{L}{\pi\hbar} \left(\frac{N^2\pi^2\hbar^2}{8mL^2}\right)^{3/2} = \frac{N^3\pi^2\hbar^2}{24\,mL^2} = \frac{N\pi^2\hbar^2}{24\,m}\left(\frac{N}{L}\right)^2$$

$$e.\ E(\tau) = \int_0^\infty d\epsilon\, n_s(\epsilon)\rho(\epsilon)\epsilon \overset{\text{proportional}}{\sim} \int_0^\infty d\epsilon \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1+e^{\beta(\mu-\epsilon)}+e^{2\beta(\mu-\epsilon)}} \cdot \epsilon^{1/2}$$

low temperature heat capacity

$$C = \frac{\partial E}{\partial \tau} = \frac{d\beta}{d\tau}\frac{\partial E}{\partial \beta} = \frac{-1}{\tau^2}\frac{\partial E}{\partial \beta}$$

$$\frac{\partial E}{\partial \beta} \sim \int_0^\infty \epsilon^{1/2} \frac{\partial}{\partial \beta} \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1+e^{\beta(\mu-\epsilon)}+e^{2\beta(\mu-\epsilon)}}$$

$\alpha = 0?$ $\frac{1}{2}?$

alpha = 1, like for Fermions.
Use either the Sommerfeld expansion, or the heuristic derivation.