Jan 2000 #1 (QM)

central potential $V(r)$: spherically symmetric

Backward scattering: $\theta = \pi$

a. $\left.\frac{d\sigma}{d\Omega}\right|_{back} = A \frac{\exp(-4\lambda k)}{k^2}$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}'}V(r')d^3\vec{r}' \qquad \vec{k}_f = k\hat{r} \quad \vec{k}_i = k\hat{z}$$

Because of the spherical symmetry of the potential, we can align the $\vec{r}'$ coordinate system so that the z' axis is parallel to $\vec{q} \equiv \vec{k}_f - \vec{k}_i$

$$\vec{q}\cdot\vec{r}' = qr'\cos\theta'$$

$$|q|^2 = \vec{q}\cdot\vec{q} = (k^2)(\hat{r}-\hat{z})\cdot(\hat{r}-\hat{z}) = k^2(1-2\hat{r}\cdot\hat{z}+1)$$
$$= 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$$
$$\Rightarrow |q| = 2k\sin\frac{\theta}{2}$$: only depends on θ

$$f(\theta,\phi) = \frac{-m}{\hbar^2}\iint_{-1}^{1} dr'\,r'^2\,d(\cos\theta')\,V(r')\,e^{-iqr'\cos\theta'}$$

θ' Integral: $\frac{1}{-iqr'}\left[e^{-iqr'} - e^{+iqr'}\right] = \frac{2\sin qr'}{qr'}$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int dr'\,r'\sin(qr')\,V(r')$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2}{\hbar^2\cdot 4k^2\sin^2\frac{\theta}{2}}\cdot\left[\int dr'\,r'\sin\left(2kr'\sin\frac{\theta}{2}\right)V(r')\right]^2$$

at $\theta=\pi$, $\frac{d\sigma}{d\Omega} \sim \frac{1}{k^2}\cdot\left[\int(\quad)\right]^2 \longleftrightarrow \frac{A}{k^2}\exp(-4\lambda k)$

To go from $\theta = \pi$ to $\theta =$ arbitrary, let
$k \to k\sin\frac{\theta}{2}$

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = \frac{A}{k^2\sin^2\frac{\theta}{2}}\exp\left(-4\lambda k\sin\frac{\theta}{2}\right)}$$

b. $$\frac{m^2}{4\hbar^4\pi^2}\left|\int d^3\vec{r}\,' e^{-i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\right|^2 = \frac{A}{k^2}\exp[-4\lambda k]$$

$$\Rightarrow \frac{m}{2\hbar^2\pi}\left|\int d^3\vec{r}\,' e^{-i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\right| = \frac{\sqrt{A}}{k}\exp(-2\lambda k)$$

$|q| = 2|k|$

$$\Rightarrow \int d^3\vec{r}\,' e^{-i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,') = \frac{4\hbar^2\pi\sqrt{A}}{m}\,\frac{1}{q}\exp(-\lambda q)$$

'Fourier Transform

Inverse Transform:

$$V(\vec{r}) = \frac{1}{(2\pi)^3}\cdot\frac{4\pi\hbar\sqrt{A}}{m}\int d^3\vec{q}\, e^{i\vec{q}\cdot\vec{r}}\,\frac{1}{q}\exp(-\lambda q)$$

orient $\vec{q}$ coord system so $\hat{z}$ is along $\hat{r}$ ...

$$V(\vec{r}) = \frac{\hbar\sqrt{A}}{\pi m}\int_0^\infty dq\, q\exp(-\lambda q)\underbrace{\int_{-1}^{1} d(\cos\theta)\, e^{iqr\cos\theta}}_{\frac{1}{iqr}\left(e^{iqr}-e^{-iqr}\right)}$$

$$V(\vec{r}) = \frac{\hbar\sqrt{A}}{i\pi m r}\int_0^\infty dq\; e^{q(-\lambda+ir)} - e^{q(-\lambda-ir)}$$

$$= \frac{1}{i}\left[\frac{-1}{-\lambda+ir} + \frac{1}{-\lambda-ir}\right] = \frac{1}{i}\left[\frac{-\lambda+ir - (-\lambda-ir)}{\lambda^2+r^2}\right] = \frac{1}{i}\left[\frac{2ir}{\lambda^2+r^2}\right] = \frac{2r}{\lambda^2+r^2}$$

$$V(r) = \frac{2\hbar A^{1/2}}{\pi m}\,\frac{1}{\lambda^2+r^2}$$