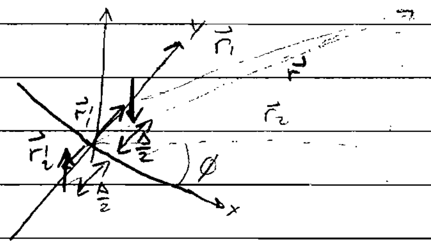
Jan 2000 # 3 (EM)



$$\vec{r}_1 = \vec{r} - \vec{r}_1'$$

$$|\vec{r}_1|^2 = |\vec{r} - \vec{r}_1'|^2 = r^2 - 2\vec{r}\cdot\vec{r}_1' + r_1'^2 \approx r^2 - 2\vec{r}\cdot\vec{r}_1'$$

$$r_1 = r\left(1 - \frac{2}{r^2}\vec{r}\cdot\vec{r}_1\right)^{1/2}$$

$$r_1 \approx r\left(1 - \frac{1}{r^2}\vec{r}\cdot\vec{r}_1\right) = r - \hat{r}\cdot\vec{r}_1$$

Long wavelength, far field approximation

$d \ll \lambda \ll r$

$\Delta = \frac{\lambda}{2}$

Treat the angles from the two dipoles as equal; to lowest order only treat the phases in e^{ikr} differently

Each dipole: center fed:

assume charge distribution is uniform $\frac{I}{\circ}$

$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

$$\frac{\partial J}{\partial z} = i\omega\rho \qquad \frac{\partial I}{\partial z} = i\omega A\rho = i\omega\mu \qquad \mu = \text{charge/unit length} = \text{constant } (z>0)$$

$$I(z) = i\omega\mu z + \text{Const.} \qquad (z>0)$$

$$I(0) = I_0 \qquad I(z) = i\omega\mu z + I_0$$

$$I\left(\tfrac{d}{2}\right) = 0 \Rightarrow I_0 + i\omega\mu\frac{d}{2} = 0 \qquad \mu = \frac{-I_0}{i\omega}\frac{2}{d} \qquad (z>0)$$

opposite for $z<0$

$$\text{Dipole} \quad \vec{p} = \int \rho\,\vec{x}\,d^3\vec{x} \Rightarrow p_z = 2\int_0^{d/2} \mu z\,dz = \mu\left(\tfrac{d}{2}\right)^2 = \frac{iI_0}{\omega}\frac{d}{2} = \frac{idI_0}{2\omega}$$

$$r_1 \approx r - \hat{r}\cdot\vec{r}_1' \qquad \vec{r}_1' = +\frac{\Delta}{2}\hat{y} \qquad \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$r_2 \approx r - \hat{r}\cdot\vec{r}_2' \qquad \vec{r}_2' = -\frac{\Delta}{2}\hat{y}$$

$$\Rightarrow r_1 \approx r - \frac{\Delta}{2}\sin\theta\sin\phi$$

$$r_2 \approx r + \frac{\Delta}{2}\sin\theta\sin\phi$$

Field of each dipole: $\vec{A}(\vec{x}) = -ik\vec{p}\,\frac{e^{ikr}}{r}$

$$\vec{B} = \nabla\times\vec{A} = ik\hat{n}\times\vec{A} \qquad (\text{ignoring } \tfrac{1}{r^2} \text{ terms})$$

$$= k^2\,\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r} \qquad \vec{E} = -\hat{n}\times\vec{B}$$

Putting time dependence back in,

$$\vec{B}_1 = k^2 \hat{n}\times\vec{p}\,\frac{e^{i(kr_1-\omega t)}}{r} = k^2\hat{n}\times\vec{p}\,\frac{e^{i(kr-\omega t)}}{r}\,e^{-ik\frac{\Delta}{2}\sin\theta\sin\phi}$$

$$\vec{B}_2 = k^2\hat{n}\times\vec{p}\,\frac{e^{i(kr_2-\omega t-\pi)}}{r} = -k^2\hat{n}\times\vec{p}\,\frac{e^{i(kr-\omega t)}}{r}\,e^{ik\frac{\Delta}{2}\sin\theta\sin\phi}$$

$$\vec{B}_{tot} = -k^2\hat{n}\times\vec{p}\left(e^{ik\frac{\Delta}{2}\sin\theta\sin\phi} - e^{-ik\frac{\Delta}{2}\sin\theta\sin\phi}\right)\frac{e^{i(kr-\omega t)}}{r}$$

$$= -2i\,k^2\hat{n}\times\vec{p}\,\sin\left(k\frac{\Delta}{2}\sin\theta\sin\phi\right)\frac{e^{i(kr-\omega t)}}{r}$$

$$\vec{E}_{tot} = -\hat{n}\times\vec{B}$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = r^2\hat{n}\cdot\langle\vec{S}\rangle = \frac{c}{8\pi}\,\mathrm{Re}\left(r^2\hat{n}\cdot(\vec{E}\times\vec{B}^*)\right)$$

$$\vec{E}\times\vec{B}^* = (-\hat{n}\times\vec{B})\times\vec{B}^* = B^*\times(\hat{n}\times\vec{B}) = \hat{n}|\vec{B}|^2$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{c}{8\pi}r^2|B|^2$$

$$= \frac{c}{8\pi}\cdot 4k^4\,|\hat{n}\times\vec{p}|^2\sin^2\left(\frac{k\Delta}{2}\sin\theta\sin\phi\right) \qquad \vec{p}\sim\hat{z} \qquad \hat{n}\times\hat{z}\sim\sin\theta$$

$$= \frac{ck^4}{2\pi}\cdot|p|^2\sin^2\theta\,\sin^2\left(\frac{k\Delta}{2}\sin\theta\sin\phi\right) \qquad k=\frac{\omega}{c} \qquad k=\frac{2\pi}{\lambda}\quad \Delta=\frac{\lambda}{2}$$

$$p=\frac{idI_0}{2\omega} \qquad k\Delta = \pi$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\omega^2 d^2 I_0^2}{8\pi c^3}\sin^2\theta\,\sin^2\left[\frac{\pi}{2}\sin\theta\sin\phi\right]$$