2003 Part I Q6 Asymptotics

$W(x) = \int_0^\infty e^{-(t+\frac{x}{t})} dt$

$\phi = -(t + \frac{x}{t})$

$\phi' = -(1 - \frac{x}{t^2})$ saddle at $t_s = \sqrt{x}$

$\phi'' = \frac{-2x}{t^3}$

Variable normalization so that $|\phi''| \to \infty$

$\to$ let $t = \sqrt{x}\, s$ $dt = \sqrt{x}\, ds$

$W = \sqrt{x} \int_0^\infty ds\, e^{-(\sqrt{x} s + \frac{\sqrt{x}}{s})} = \sqrt{x} \int_0^\infty ds\, e^{-\sqrt{x}(s+\frac{1}{s})}$

$\phi = -\sqrt{x}(s + \frac{1}{s})$

$\phi' = -\sqrt{x}(1 - \frac{1}{s^2})$ saddle at $s=1$

$\phi'' = -\frac{2\sqrt{x}}{s^3}$ $\phi''(1) = -2\sqrt{x}$ $\phi(1) = -2\sqrt{x}$

Endpoint contribution? At $s \to 0$, $\phi \to -\infty$ no contribution

$W \sim \sqrt{x} e^{-2\sqrt{x}} \int_{1-\infty}^{1+\infty} ds\, e^{-\sqrt{x} s^2}$

$W \sim 2\sqrt{x} e^{-2\sqrt{x}} \underbrace{\int_1^{1+\infty} ds\, e^{-\sqrt{x}(s-1)^2}}_{u = \sqrt{x}(s-1)^2}$

$du = 2\sqrt{x}(s-1) ds$ $s - 1 = \left(\frac{u}{\sqrt{x}}\right)^{1/2} = \frac{u^{1/2}}{x^{1/4}}$

$du = \frac{2x^{1/2}u^{1/2}}{x^{1/4}} ds = 2x^{1/4} u^{1/2} ds$

$ds = \frac{du}{2x^{1/4} u^{1/2}}$

$W \sim \frac{2x^{1/2} e^{-2\sqrt{x}}}{2x^{1/4}} \underbrace{\int_0^\infty du \frac{e^{-u}}{u^{1/2}}}_{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$

$\boxed{W \sim \pi^{1/2} x^{1/4} e^{-2x^{1/2}}}$