2003 Part II Q1A

Comp

1. $\frac{\partial F(U)}{\partial x} = A\frac{\partial U}{\partial x}$ $\qquad U_j^* = U_j^n - \frac{\delta t}{\delta x}A\left[U_j^n - U_{j-1}^n\right]$

$$\Rightarrow U_j^{n+1} = \frac{1}{2}\Big\{U_j^n + U_j^n - \frac{\delta t}{\delta x}A\left[U_j^n - U_{j-1}^n\right] - \frac{\delta t}{\delta x}A\left[U_j^n - \frac{\delta t}{\delta x}A\left[U_j^n - U_{j-1}^n\right] - U_{j-1}^n + \frac{\delta t}{\delta x}A\left[U_{j-1}^n - U_{j-2}^n\right]\right] - \frac{\delta t}{\delta x}A\left[U_j^n - 2U_{j-1}^n + U_{j-2}^n\right]\Big\}$$

$$U_j^{n+1} = U_j^n - \frac{\delta t}{2\delta x}A\left[3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right] + \frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2\left[U_j^n - 2U_{j-1}^n + U_{j-2}^n\right]$$

Truncation error:

$$U_j^{n+1} = U_j^n + \delta t\frac{\partial U}{\partial t} + \frac{1}{2!}\delta t^2\frac{\partial^2 U}{\partial t^2} + \frac{1}{6}\delta t^3\frac{\partial^3 U}{\partial t^3} + \dots$$

$$U_{j-1}^n = U_j^n - \delta x\frac{\partial U}{\partial x} + \frac{1}{2}\delta x^2\frac{\partial^2 U}{\partial x^2} - \frac{1}{6}\delta x^3\frac{\partial^3 U}{\partial x^3} + \frac{1}{24}\delta x^4\frac{\partial^4 U}{\partial x^4} + \dots$$

$$U_{j-2}^n = U_j^n - 2\delta x\frac{\partial U}{\partial x} + 2\delta x^2\frac{\partial^2 U}{\partial x^2} - \frac{4}{3}\delta x^3\frac{\partial^3 U}{\partial x^3} + \frac{2}{3}\delta x^4\frac{\partial^4 U}{\partial x^4} + \dots$$

$$\frac{U_j^{n+1} - U_j^n}{\delta t} = \frac{\partial U}{\partial t} + \frac{1}{2}\delta t\frac{\partial^2 U}{\partial t^2} + \frac{1}{6}\delta t^2\frac{\partial^3 U}{\partial t^3} + \dots$$

$$\frac{3U_j^n - 4U_{j-1}^n + U_{j-2}^n}{2\delta x} = \frac{\partial U}{\partial x} - \frac{1}{3}\delta x^2\frac{\partial^3 U}{\partial x^3} + \dots$$

$$\frac{U_j^n - 2U_{j-1}^n + U_{j-2}^n}{\delta x^2} = \frac{\partial^2 U}{\partial x^2} - \delta x\frac{\partial^3 U}{\partial x^3} + \dots$$

Plugging these in, we have

$$\frac{\partial U}{\partial t} + \frac{1}{2}\delta t\frac{\partial^2 U}{\partial t^2} + \frac{1}{6}\delta t^2\frac{\partial^3 U}{\partial t^3} + \dots = A\frac{\partial U}{\partial x} - \frac{1}{3}\delta x^2 A\frac{\partial^3 U}{\partial x^3} + A^2\frac{1}{2}\delta t\frac{\partial^2 U}{\partial x^2} - A^2\frac{1}{2}\delta x\,\delta t\frac{\partial^3 U}{\partial x^3} + \dots$$

$$\frac{\partial U}{\partial t} = -A\frac{\partial U}{\partial x} \Rightarrow \frac{\partial^2 U}{\partial t^2} = -A\frac{\partial}{\partial x}\frac{\partial U}{\partial t} = A^2\frac{\partial^2 U}{\partial x^2}$$

Cancellation of $O(\delta t)$ term

$\Rightarrow$ 2nd order accurate in time & space

2. Let $s = \frac{\delta t}{\delta x} A$ $\quad U_j^n \to r^n \tilde{U}_k e^{ij\theta}$

$$r = 1 - \frac{s}{2}\left[3 - 4e^{-i\theta} + e^{-2i\theta}\right] + \frac{s^2}{2}\left[1 - 2e^{-i\theta} + e^{-2i\theta}\right]$$

Require $|r| < 1$ for stability. Check at $\theta = 0, \frac{\pi}{2}, \pi$ {this always works as far as I know}

at $\theta = 0$: $r = 1 - \frac{s}{2}[3 - 4 + 1] + \frac{s^2}{2}[1 - 2 + 1)$ $\qquad r = 1$

at $\theta = \frac{\pi}{2}$: $r = 1 - \frac{s}{2}[3 + 4i - 1] + \frac{s^2}{2}[1 + 2i - 1] = 1 - \frac{s}{2}[2 + 4i] + s^2 i$

$r = 1 - s + is(s-2)$

$|r|^2 = (1-s)^2 + s^2(s-2)^2$

$|r|^2 < 1$: $\quad 1 - 2s + s^2 + s^4 - 4s^3 + 4s^2 < 1$

$s^3 - 4s^2 + 5s - 2 < 0$

$\Rightarrow (s-1)(s-1)(s-2) < 0 \Rightarrow (s-1)^2(s-2) < 0$

$\to s < 2$ $\quad$ or $\quad \delta t < \frac{2\delta x}{A}$

at $\theta = \pi$: $\quad r = 1 - \frac{s}{2}[3 + 4 + 1] + \frac{s^2}{2}[1 + 2 + 1]$

$r = 1 - 4s + 2s^2$

$r < 1$: $\quad 1 - 4s + 2s^2 < 1$

$2s^2 - 4s < 0$

$s < 2$ $\qquad$ <u>same condition</u>

$$\boxed{\delta t < \frac{2\delta x}{A}}$$

3. advantages: explicit method, simple. 2^{nd} order accurate

disadvantages: For stability, a small timestep is required at small δx.

But: at least it can be stable, as compared with FTCS which is unstable