2013 Irreversibles Problem (1-4)

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For this problem, we will take the limit that $m_i/m_e \to \infty$, so that we may use the Lorentz collision operator and also assume the ions are stationary, so that electron flow is equivalent to current. There are two different timescales in this problem, the gyration timescale and the collisional timescale. First we will find the perturbed distribution function on the gyration timescale, and then we will use that to find the perturbed distribution on the collisional timescale. A scalar cross-field (Pedersen) conductivity will only exist on the collisional timescale, since on the cyclotron time scale $\mathbf{E}\times\mathbf{B}$ motion will drive the off-diagonal Hall conductivity. For reference, the ordering I'm assuming is:

$$\frac{Ec}{Bv_{The}} \sim \frac{\nu}{\Omega} \ll 1 \tag{1}$$

Start by assuming homogeneity and steady state, so that the set of equations becomes:

$$\Omega\frac{\partial f_0}{\partial\theta} = 0 \tag{2}$$

$$\Omega\frac{\partial f_1}{\partial\theta} = \frac{e}{m_e}\mathbf{E}\cdot\frac{\partial f_0}{\partial\mathbf{v}} + C[f_0] \tag{3}$$

$$\Omega\frac{\partial f_2}{\partial\theta} = \frac{e}{m_e}\mathbf{E}\cdot\frac{\partial f_1}{\partial\mathbf{v}} + C[f_1] \tag{4}$$

Now, the first equation simply says that f_0 is gyrotropic, but we want to look near thermal equilibrium (assume the plasma has been around for many collision times), so take it to be Maxwellian, $f_0 = f_M$. Since $C[f_0]$ is then 0, we can get f_1 by integrating the second equation.

$$f_1 = \frac{e}{m_e\Omega}\int d\theta\,\frac{\partial f_M}{\partial\mathbf{v}}\cdot\mathbf{E} \tag{5}$$

$$= -\frac{e}{v_{The}^2 m_e\Omega}\int d\theta\,\mathbf{v}\cdot\mathbf{E}f_M \tag{6}$$

$$= -\left(\frac{eE}{\rho_e m_e\Omega^2}\right)\left(\frac{v}{v_{The}}\right)\sin\phi f_M\int d\theta\,\cos\theta \tag{7}$$

$$= -\frac{E}{e}\left(\frac{\omega_{pe}^2}{4\pi n_0\Omega}\right)\left(\frac{v}{v_{The}^2}\right)f_M\sin\phi\sin\theta \tag{8}$$

$$= i\frac{E}{e}\left(\frac{\omega_{pe}^2}{8\pi n_0\Omega}\right)\left(\frac{v}{v_{The}^2}\right)\left(Y_1^1 - Y_1^{-1}\right)f_M \tag{9}$$

$$\tag{10}$$

Where I've assumed $\mathbf{E}$ points in the x-direction. Note that this describes a flow perpendicular to the applied electric field, as expected of an $\mathbf{E}\times\mathbf{B}$ flow. We can now solve for f_2.

Using the Lorentz collision operator we find:

$$-C[f_1] = 2\nu_{ei}\left(\frac{v_{The}^3}{v^3}\right)f_1 \tag{11}$$

and so:

$$\Omega\frac{\partial f_2}{\partial\theta} = \frac{e}{m_e}(\mathbf{E}\cdot\mathbf{v})\left(\frac{1}{v^2} - \frac{1}{v_{The}^2}\right)f_1 - 2\nu_{ei}\left(\frac{v_{The}^3}{v^3}\right)f_1 \tag{12}$$

$$\Omega\frac{\partial f_2}{\partial\theta} = \frac{eE}{m_e}(\sin\phi\cos\theta)\left(\frac{1}{v} - \frac{v}{v_{The}^2}\right)f_1 - 2\nu_{ei}\left(\frac{v_{The}^3}{v^3}\right)f_1 \tag{13}$$

Now the first term on the right hand side is nonlinear in $\mathbf{E}$ so we will ignore it. As a side note, it goes as $\sin\theta\cos\theta$, and so when integrated over θ will give a $\sin^2\theta$ term, so this term cannot drive flows parallel to the applied field. Then, the equation for f_2 is simple:

$$\frac{\partial f_2}{\partial\theta} = -2\frac{\nu_{ei}}{\Omega}\left(\frac{v_{The}^3}{v^3}\right)f_1 \tag{14}$$

$$f_2 = E\left(\frac{2e\rho_e}{m_e v^2}\right)\left(\frac{\nu_{ei}}{\Omega}\right)f_M \sin\phi\int d\theta\,\sin\theta \tag{15}$$

$$f_2 = -E\left(\frac{\omega_{pe}^2\rho_e}{2\pi e n_0 v^2}\right)\left(\frac{\nu_{ei}}{\Omega}\right)(\sin\phi\cos\theta)f_M \tag{16}$$

So now we can compute the perturbed current parallel to the applied field (or equivalently the scalar cross-field conductivity).

$$\sigma_P = -\frac{e}{E}\left[\int d^3\mathbf{v}\,\mathbf{v}f_2\right]_x \tag{17}$$

$$=\left(\frac{\omega_{pe}^2\rho_e}{2\pi n_0}\right)\left(\frac{\nu_{ei}}{\Omega}\right)\iiint d\theta d\phi dv\, v\sin^3\phi\cos^2\theta f_M \tag{18}$$

$$=\left(\frac{2\omega_{pe}^2\rho_e}{3n_0}\right)\left(\frac{\nu_{ei}}{\Omega}\right)\int dv\, v f_M \tag{19}$$

$$=\frac{1}{3\sqrt{2}\pi^{3/2}}\left(\frac{\omega_{pe}^2}{\Omega}\right)\left(\frac{\nu_{ei}}{\Omega}\right) \tag{20}$$

From Krommes' notes, we know that for the Lorentz operator, $\nu_{ei}^{-1} \neq \tau_e$. Instead, we have $\nu_{ei} = 3\sqrt{2\pi}/4\tau_e$ (equation 26.50). So putting this in gives:

$$\sigma_P = \frac{1}{4\pi\tau_e}\left(\frac{\omega_{pe}}{\Omega}\right)^2 \tag{21}$$

This is just the highest-order approximation to the Pedersen conductivity, which is what we expected. Now in the electron fluid equation, the friction force must balance the applied electric field in steady state, as well as the induced $\mathbf{U_e}\times\mathbf{B}$ force so we must first compute the Hall conductivity. To highest order this is contained in f_1.

$$\sigma_H = -\frac{e}{E}\left[\int d^3\mathbf{v}\,\mathbf{v}f_1\right]_y \tag{22}$$

$$=\left(\frac{1}{4\pi\rho_e n_0 v_{The}}\right)\left(\frac{\omega_{pe}}{\Omega}\right)^2\iiint d\theta d\phi dv\, v^4\sin^3\phi\sin^2\theta f_M \tag{23}$$

$$=\left(\frac{1}{3\rho_e n_0 v_{The}}\right)\left(\frac{\omega_{pe}}{\Omega}\right)^2\int dv\, v^4 f_M \tag{24}$$

$$=\frac{1}{4\pi}\left(\frac{\omega_{pe}^2}{\Omega}\right) \tag{25}$$

So now we can put it all together.

$$
\begin{aligned}
R_x =& en_e\Big(\mathbf{E} + \frac{1}{c}\mathbf{U_{e\perp}} \times \mathbf{B}\Big)_x && (26)\\
=& \frac{en_e}{\sigma_P}J_x - \frac{BJ_y}{c} && (27)\\
=& \Big(\frac{en_e}{\sigma_P} - \frac{B\sigma_H}{c\sigma_P}\Big)J_x && (28)\\
=& -\frac{(mn)_e}{\tau_e}\Big(\big(\tau_e\Omega\big)^2 - (\tau_e\Omega)^2\Big)\big(\mathbf{u}_{e,\perp} - \mathbf{u}_{i,\perp}\big) && (29)\\
=& 0 && (30)
\end{aligned}
$$

So really, what we've found is that the off-diagnoal Hall conductivity sets up an electron flow pattern that exactly balances the force from the electric field to highest order. So to compute the frction force we need to go to the next order. For simplicity, we will ignore terms non-linear in the electric field, so that the f_2 we computed is complete. More properly, we should include these terms, but they will lead to friction forces non-linear in the flow which we don't want to consider.

The next order equation linear in electric field is:

$$
\begin{aligned}
\Omega\frac{\partial f_3}{\partial \theta} &= C[f_2] && (31)\\
&= -2\nu_{ei}\left(\frac{v_{The}^3}{v^3}\right)f_2 && (32)\\
&= E\left(\frac{\omega_{pe}^2\nu_{ei}^2}{\pi e n_0 \Omega^2}\right)\Big(\frac{v_{The}^4}{v^5}\Big)(\sin\phi\cos\theta)f_M && (33)
\end{aligned}
$$

So:

$$
f_3 = E\left(\frac{\omega_{pe}^2\nu_{ei}^2}{\pi e n_0 \Omega^3}\right)\Big(\frac{v_{The}^4}{v^5}\Big)(\sin\phi\sin\theta)f_M \tag{34}
$$

This is problematic since it is divergent at $v = 0$, and so we cannot get a flow out of it. I'm not really sure what to do here expect appeal to a more general collision operator. If we pretend that this integral is convergent, we can get the scaling of the the next order term in σ_H:

$$
\sigma_H \sim \frac{1}{4\pi}\Big(\frac{\omega_{pe}^2}{\Omega}\Big)\Big(1 - (\tau_e\Omega)^{-2}\Big) \tag{35}
$$

Which is the scaling we should get from this approach, and is in agreement with the exact definition of the Hall conductivity. Using this, we find to lowest order and linear in $\mathbf{E}$:

$$
\mathbf{R}_\perp \sim -\frac{(mn)_e}{\tau_e}\big(\mathbf{u}_{e,\perp} - \mathbf{u}_{i,\perp}\big) \tag{36}
$$

Which is Braginskii's result, however this is still not complete since there will be corrections to σ_P which come from the next order distribution function, but are of this same order in the friction force. So really here we need to make sure this is non-vanishing, but in order to do that we need to compute coefficients, which, of course we can't do unless we have a collision operator which gives distributions with valid first moments at all orders. This either means that this problem cannot be done with the Lorentz operator, or that there is a simpler approach than the one I used. At this point, I'm just going to say that by looking at the exact forms of the Hall and Pedersen conductivities, we can see that Braginskii is correct, and this order is both non-vanishing and has a coefficient of 1 and be done with it.