Jan 2000 #2 (QM)

$$H_0 = -\frac{\hbar^2}{2m}(\nabla_1^2+\nabla_2^2) + V(\vec{r}_1)+V(\vec{r}_2)$$

$$V(\vec{r}) = \tfrac{1}{2}kr^2$$

$$H' = \epsilon\left(x_1x_2 + y_1y_2 - 2z_1z_2\right)$$

Unperturbed Hamiltonian splits into 6 independent oscillators

define $k \equiv m\omega^2$

$|\psi\rangle \to |n_{1x}n_{1y}n_{1z}n_{2x}n_{2y}n_{2z}\rangle$, $\quad E = 3\hbar\omega + \hbar\omega(n_{1x}+n_{1y}+n_{1z}+n_{2x}+n_{2y}+n_{2z})$

Ground state is nondegenerate: $|000000\rangle$

$$E_n^1 = \langle n^0|H'|n^0\rangle$$

$$E_{gs}^1 = \epsilon\langle 000000|x_1x_2|000000\rangle + \ldots$$

$$x_1 = (a_{1x}+a_{1x}^\dagger)\frac{x_0}{\sqrt{2}} \qquad x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$$

x_1 mixes the states; every term is 0 in E_{gs}^1

$$E_n^2 = \langle n^0|H'|n^1\rangle = \sum_{m\neq n}\frac{\langle n^0|H'|m^0\rangle\langle m^0|H'|n^0\rangle}{E_n^0-E_m^0} = \sum_{m\neq n}\frac{|\langle n^0|H'|m^0\rangle|^2}{E_n^0-E_m^0}$$

$$E_{gs}^2 = \sum_{n_{1x}\cdots n_{2z}\neq 0}\frac{|\langle n_{1x}n_{1y}n_{1z}n_{2x}n_{2y}n_{2z}|H'|000000\rangle|^2}{3\hbar\omega-(3\hbar\omega+\hbar\omega(n_{1x}+\cdots+n_{2z}))}$$

$$= \frac{-\epsilon^2}{\hbar\omega}\sum\frac{|\langle n_{1x}\cdots n_{2z}|x_1x_2+y_1y_2-2z_1z_2|000000\rangle|^2}{n_{1x}+n_{1y}+n_{1z}+n_{2x}+n_{2y}+n_{2z}}$$

First term: $\langle n_{1x}\cdots n_{2z}|x_1x_2|000000\rangle \qquad x_1 = \frac{x_0}{\sqrt{2}}(a_{1x}+a_{1x}^\dagger) \qquad x_2 = \frac{x_0}{\sqrt{2}}(a_{2x}+a_{2x}^\dagger)$

$a|0\rangle = 0 \qquad a^\dagger|0\rangle = |1\rangle$

$$= \frac{x_0^2}{2}\langle n_{1x}\cdots n_{2z}|100100\rangle = \frac{\hbar}{2m\omega}\cdot\delta_{1n_{1x}}\delta_{1n_{2x}}\delta_{0\text{-rest}}$$

Similarly, Second term $= \frac{\hbar}{2m\omega}\delta_{1n_{1y}}\delta_{1n_{2y}}\delta_{0\text{-rest}}$

Third term $= -2\left(\frac{\hbar}{2m\omega}\delta_{1n_{1z}}\delta_{1n_{2z}}\delta_{0\text{-rest}}\right)$

$$E_{gs}^2 = \frac{-\epsilon^2\hbar}{4m^2\omega^3}\sum\frac{|A+B-2C|^2}{n_{1x}+\cdots+n_{2z}}$$

where $A=1$ if $n_{1x}=1=n_{2x}$, the rest $=0$

$B=1$ if $n_{1y}=1=n_{2y}$, "

$C=1$ if $n_{1z}=1=n_{2z}$, the rest $=0$

$(A+B-2C)^2 = (A+B)^2 - 2(A+B)(2C) + 4C^2$

$$E_{gs}^2 = \frac{-\epsilon^2\hbar}{4m^2\omega^3}\sum_{n_i} \frac{A^2+2AB+B^2-4AC-4BC+4C^2}{n_{1x}+\cdots+n_{2z}}$$

A^2 term: $\frac{1}{2}$ 2AB term: 0; all cross terms vanish

B^2 term: $\frac{1}{2}$

$4C^2$ term: $4\cdot\frac{1}{2}=2$

$$E_{gs}^2 = \frac{-\epsilon^2\hbar}{4m^2\omega^3}\left[\frac{1}{2}+\frac{1}{2}+2\right]$$

$$\boxed{E_{gs}^2 = -\frac{3\hbar\epsilon^2}{4m^2\omega^3}}$$