2001 Part I Q1 GPPI

a. $f_e(v_z) = n\delta(v_z)$ $\quad f_i(v_z) = n\delta(v_z - u_0)$

Vlasov Equation $\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} + \frac{q\vec{E}}{m}\cdot\frac{\partial f}{\partial \vec{v}} = 0$ $\quad \vec{E}_0 = 0$

Linearized Vlasov Eqn: $\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{x}} + \frac{q\vec{E}}{m}\cdot\frac{\partial f_0}{\partial \vec{v}} = 0$

$f_1, \vec{E} \sim \exp(ikz - i\omega t)$ $\quad \vec{E} = -\nabla\phi = -ik\phi\hat{z}$

$$-i\omega f_1 + ikv_z f_1 - \frac{ikq\phi}{m}\cdot\frac{\partial f_0}{\partial v_z} = 0$$

$$(\omega - kv_z)f_1 = -\frac{kq\phi}{m}\frac{\partial f_0}{\partial v_z} \implies f_{1j} = \frac{-kq_j\phi}{m_j(\omega - kv_z)}\frac{\partial f_{0j}}{\partial v_z}$$

Poisson's Equation: $-\nabla^2\phi = 4\pi\rho_1 = 4\pi e(Zn_{i1} - n_{e1}) = 4\pi e\sum_j Z_j n_j$

$$k^2\phi = 4\pi e\sum_j Z_j\int dv_z\, f_{1j}$$

$$\int dv_z\, f_{1j} = \frac{-kq_j\phi}{m_j}\int dv_z \frac{1}{(\omega - kv_z)}\frac{\partial f_{0j}}{\partial v_z} = \frac{kq_j\phi}{m_j}\int dv_z\, f_{0j}\frac{\partial}{\partial v_z}\left(\frac{1}{\omega - kv_z}\right)$$

$$= \frac{k^2 q_j\phi}{m_j}\int dv_z\, \frac{n\delta(v_z - u_j)}{(\omega - kv_z)^2} = \frac{k^2 n q_j\phi}{m_j}\frac{1}{(\omega - ku_j)^2} \qquad u_j = \begin{cases} 0 & e^- \\ u_0 & \text{ions}\end{cases}$$

$$\implies k^2\phi = \sum_j \frac{4\pi q_j^2 n}{m_j}k^2\phi\frac{1}{(\omega - ku_j)^2}$$

$$\implies 1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{(\omega - ku_0)^2}$$

b. $\epsilon = \frac{\omega_{pi}^2}{\omega_{pe}^2} \ll 1$ $\quad$ Let $\frac{\omega}{\omega_{pe}} \equiv z$, $\frac{ku_0}{\omega_{pe}} \equiv \lambda$

Then $$1 = \frac{1}{z^2} + \frac{\epsilon}{(z - \lambda)^2}$$

• Calculate $\frac{\partial z}{\partial \lambda}$ $\left(\sim \frac{\partial \omega}{\partial k}\right)$ by implicit differentiation:

$$0 = -\frac{2}{z^3}\frac{\partial z}{\partial \lambda} - \frac{2\epsilon}{(z-\lambda)^3}\left(\frac{\partial z}{\partial \lambda} - 1\right)$$

$$\frac{\partial z}{\partial \lambda}\left(\frac{1}{z^3} + \frac{\epsilon}{(z-\lambda)^3}\right) = \frac{\epsilon}{(z-\lambda)^3} \qquad \frac{\partial z}{\partial \lambda} = \frac{\epsilon}{\epsilon + (1-\frac{\lambda}{z})^3}$$

Maximum growth rate: $\frac{\partial}{\partial \lambda}\left[\operatorname{Im} z\right] = 0 = \operatorname{Im}\frac{\partial z}{\partial \lambda} = \frac{\frac{\partial z}{\partial \lambda} - \left(\frac{\partial z}{\partial \lambda}\right)^*}{2i}$

$\Rightarrow$ This condition yields $\left(1 - \frac{\lambda}{z}\right)^3 = \left(1 - \frac{\lambda}{z^*}\right)^3$

which is satisfied if z is of the form

$$z = \frac{\lambda}{1+\alpha e^{in\pi/3}}$$ where α is some real number, n is some integer

Substituting this form into the dispersion relation,

$$1 = \frac{(1+\alpha e^{in\pi/3})^2}{\lambda^2} + \frac{\epsilon\,(1+\alpha e^{in\pi/3})^2}{\lambda^2 \alpha^2 e^{i2n\pi/3}}$$

$$\lambda^2 = \left(1 + 2\alpha e^{in\pi/3} + \alpha^2 e^{2in\pi/3}\right) + \epsilon\left(1 + \frac{2}{\alpha}e^{-in\pi/3} + \frac{1}{\alpha^2}e^{-i2n\pi/3}\right)$$

Consider the imaginary part of this eqn:

$$0 = 2\alpha \sin\frac{n\pi}{3} + \alpha^2 \sin\frac{2n\pi}{3} - \epsilon\left(\frac{2}{\alpha}\sin\frac{n\pi}{3} + \frac{1}{\alpha^2}\sin\frac{2n\pi}{3}\right)$$

$\Rightarrow$ Take α small (this is correct)

$$\Rightarrow \quad 2\alpha \sin\frac{n\pi}{3} \approx \epsilon \frac{1}{\alpha^2}\sin\frac{2n\pi}{3}$$

• Take $n=1$. $\Rightarrow$ $2\alpha^3\frac{\sqrt{3}}{2} = -\epsilon\frac{\sqrt{3}}{2}$ $\Rightarrow$ $\alpha^3 = -\frac{\epsilon}{2}$

$$\alpha = -\left(\frac{\epsilon}{2}\right)^{1/3}$$

$$\operatorname{Im} z = \operatorname{Im}\left(\frac{\lambda}{1+\alpha\cos\frac{\pi}{3} + i\alpha\sin\frac{\pi}{3}}\right) = \operatorname{Im}\left(\frac{\lambda\left[1+\alpha\cos\frac{\pi}{3} - i\alpha\sin\frac{\pi}{3}\right]}{(1+\alpha\cos\frac{\pi}{3})^2 + \alpha^2\sin^2\frac{\pi}{3}}\right)$$

$$= \frac{-\lambda\alpha\sin\frac{\pi}{3}}{1+2\alpha\cos\frac{\pi}{3}+\alpha^2}$$ $\Rightarrow$ since α is small,

$$\operatorname{Im} z \approx -\lambda\alpha\sin\frac{\pi}{3} \approx \lambda\left(\frac{\epsilon}{2}\right)^{1/3}\frac{\sqrt{3}}{2}$$

$$\Rightarrow (\operatorname{Im}\omega)_{max} \approx kU_0\left(\frac{\epsilon}{2}\right)^{1/3}\frac{\sqrt{3}}{2}$$