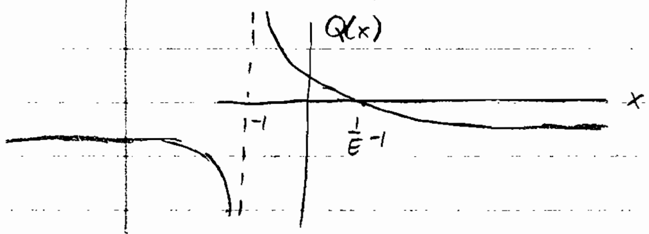
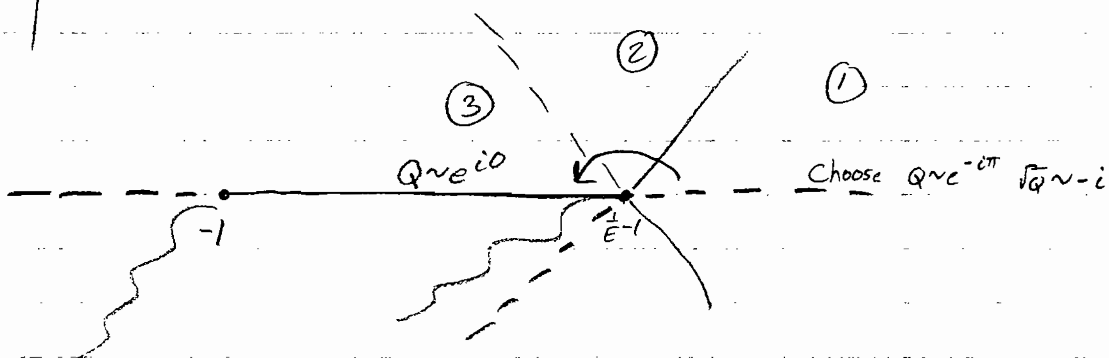
2001 Part I Q2 Asymptotics

$y'' - Ey + \frac{1}{1+x}y = 0 \qquad y'' + \left(\frac{1}{1+x} - E\right)y \equiv 0 \qquad y(0)=0, \; y(\infty)=0, \; E<1$

$Q = \frac{1}{1+x} - E$

first order pole at -1 first order zero at $\frac{1}{E} - 1 \equiv z_0$





$y(0)=0, \; y(\infty)=0$

Start in region ①, $x \to \infty$: BC's say we must have only the subdominant sol'n

$(z, z_0) \sim e^{-i\int_{z_0}^{z}\sqrt{Q}\,dx} \qquad \sqrt{Q} \sim -i \quad \Rightarrow \quad e^{-\int_{z_0}^{z}\sqrt{|Q|}\,dx}$ subdominant.

① $(z, z_0)_s$

② $(z, z_0)_d$

③ $(z, z_0)_d + i(z_0, z)_s$ using $T = i$ for (isolated) zero

"Reconnect" to 0: $(z,0)[0,z_0] + i\,[z_0,0](0,z)$

$[0, z_0] = e^{i\int_0^{z_0}\sqrt{Q}\,dx} = e^{iW}$ $W>0$ since $Q \sim e^{i0}$ here

$y = e^{iW}(z,0) + ie^{-iW}(0,z)$

At $z=0$, $(z,0) \sim Q^{-1/4}(0)$, $(0,z) \sim Q^{-1/4}(0)$

$y(0)=0 \Rightarrow e^{iW} + ie^{-iW} = 0$

$e^{i2W} = -i = e^{-i\frac{3\pi}{2}}$

$\Rightarrow 2W = \frac{3\pi}{2} + 2n\pi \qquad n=0,1,\ldots$

$W = \frac{3\pi}{4} + n\pi \qquad W = (n + \frac{3}{4})\pi \qquad n = 0, 1, 2, \ldots$

$$\boxed{W = \int_0^{\frac{1}{E}-1} \sqrt{\frac{1}{1+x} - E}\, dx = (n + \frac{3}{4})\pi} \qquad n = 0, 1, 2, \ldots$$