2000 Part II Q2 — Asymptotics

$$\Psi(t) = t^2\int_0^1 dw \left(\frac{w}{1-w}\right)^{i\lambda} e^{-4i\lambda w t^2}$$

$$= t^2\int_0^1 dw\, e^{i\lambda \ln\frac{w}{1-w}}\, e^{-4i\lambda w t^2} = t^2\int_0^1 dw\, e^{\phi}$$

$$\phi = i\lambda\left[\ln w - \ln(1-w) - 4wt^2\right]$$

$$\phi' = i\lambda\left[\frac{1}{w} + \frac{1}{1-w} - 4t^2\right]$$

$$\phi'' = i\lambda\left[-\frac{1}{w^2} + \frac{1}{(1-w)^2}\right]$$

Look for saddles: $\phi' = 0$ $\quad \frac{1}{w} + \frac{1}{1-w} - 4t^2 = 0$

$$(1-w) + w - 4t^2(w)(1-w) = 0$$

$1 - 4t^2 w + 4t^2 w^2 = 0 \qquad w^2 - w + \frac{1}{4t^2} = 0$

$w_\pm = \dfrac{1 \pm \sqrt{1-\frac{1}{t^2}}}{2} \qquad 0 < w_- < \frac{1}{2} \quad$ (for $t>1$)

$\frac{1}{2} < w_+ < 1$

$$\phi''\big|_{w\pm} : i\lambda\left[\frac{-1}{\frac{1}{4}\left(1+1-\frac{1}{t^2} \pm 2\sqrt{1-\frac{1}{t^2}}\right)} + \frac{1}{\frac{1}{4}\left(1+1-\frac{1}{t^2} \mp 2\sqrt{1-\frac{1}{t^2}}\right)}\right]$$

$$= i\lambda 4\left[\frac{1}{2-\frac{1}{t^2} \mp 2\sqrt{1-\frac{1}{t^2}}} - \frac{1}{2-\frac{1}{t^2} \pm 2\sqrt{1-\frac{1}{t^2}}}\right]$$

$$\phi''\big|_{w\pm} = 4i\lambda\left[\frac{2-\frac{1}{t^2} \pm 2\sqrt{1-\frac{1}{t^2}} - \left(2-\frac{1}{t^2} \mp 2\sqrt{1-\frac{1}{t^2}}\right)}{\left(2-\frac{1}{t^2}\right)^2 - 4\left(1-\frac{1}{t^2}\right)}\right] = 4i\lambda\,\frac{\left[\pm 4\sqrt{1-\frac{1}{t^2}}\right]}{4+\frac{1}{t^4}-\frac{4}{t^2}-4+\frac{4}{t^2}}$$

$$\phi''\big|_{w\pm} = \pm 16 i\lambda t^4\sqrt{1-\frac{1}{t^2}}$$

saddle direction: $\phi''\Delta w^2 \sim -1 \quad \Longrightarrow \pm i\sqrt{1-\frac{1}{t^2}}\,\Delta w^2 \sim -1$

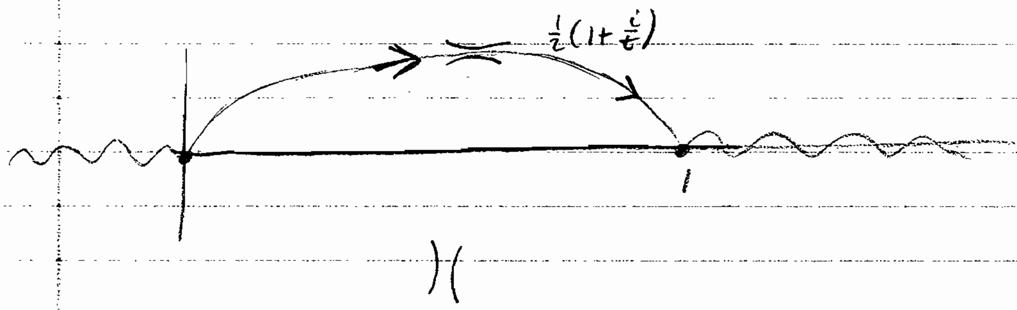
For $t \ll 1$, $\sqrt{1-\frac{1}{t^2}} = i\sqrt{\frac{1}{t^2}-1}$

$\longrightarrow \pm i^2 \Delta w^2 \sim -1 \qquad \pm\Delta w^2 \sim 1 \qquad$ At w_+, $\Delta w \sim \pm 1$ $\quad$ At w_-, $\Delta w \sim \pm i$

$$w_\pm = \frac{1 \pm i\sqrt{\frac{1}{t^2}-1}}{2} \approx \frac{1}{2} \pm \frac{i}{2t} = \frac{1}{2}\left(1 \pm \frac{i}{t}\right)$$

No endpoint contribution: $e^{i\lambda \ln w} = \cos(\lambda \ln w) + i \sin(\lambda \ln w)$, similarly at $1-w$

For $|w| \to 0$, the variation is so rapid that there is no net contribution



Note, $1 - w_\pm = w_\mp$

$\phi(w_+) = i\lambda\left[\ln w_+ - \ln(1-w_+) - 4w_+ t^2\right]$

$\phi(w_+) = i\lambda\left[\ln w_+ - \ln w_-\right] \quad - i\lambda 4 w_+ t^2$

$\ln w_+ - \ln w_- = \ln|w_+| + i\,\mathrm{Arg}(w_+) - \ln|w_-| - i\,\mathrm{Arg}(w_-)$

$\quad = i\left[\mathrm{Arg}(w_+) - \mathrm{Arg}(w_-)\right]$

But $\mathrm{Arg}(w_-) = -\mathrm{Arg}(w_+)$

$\to \quad 2i\,\mathrm{Arg}(w_+)$

As $t \to 0$, $\quad \mathrm{Arg}(w_t) \longrightarrow \frac{\pi}{2}$

$\ln w_+ - \ln w_- \to i\pi$

$\phi(w_+) \to i\lambda(i\pi) - i\lambda 4\left(\frac{1}{2}\right)\left(1+\frac{i}{t}\right)t^2$

$\quad = -\lambda\pi - 2i\lambda(t^2 + it)$

$\quad = \lambda(2t) - \lambda\pi - i2\lambda t^2$

saddle contribution: $t^2 e^{\phi_0} \int_{w_+ - \infty}^{w_+ + \infty} dw \; e^{\frac{1}{2}\phi''(w-w_+)^2} = 2t^2 e^{\phi_0} \int_{w_+}^{w_+ + \infty} dw \; e^{\frac{1}{2}\phi''(w-w_+)^2}$

$u = -\frac{\phi''(w-w_+)^2}{2} \qquad du = -\phi''(w-w_+)\,dw \qquad w - w_+ = \sqrt{\frac{2u}{-\phi''}}$

$du = \sqrt{-\phi''}\sqrt{2u}\,dw$

$\frac{2t^2 e^{\phi_0}}{\sqrt{2}\sqrt{-\phi_0''}} \int_0^\infty \frac{du}{\sqrt{u}} e^{-u} = t^2 \sqrt{\frac{2\pi}{-\phi_0''}}\, e^{\phi_0} \qquad \phi_0'' = -16\lambda t^4 \sqrt{\frac{1}{t^2} - 1} \approx -16\lambda t^3$

For $t \ll 1$, $\psi \sim t^2 \sqrt{\frac{2\pi}{16\lambda t^3}}\, e^{2\lambda t} e^{-\lambda\pi} e^{-i2\lambda t^2}$

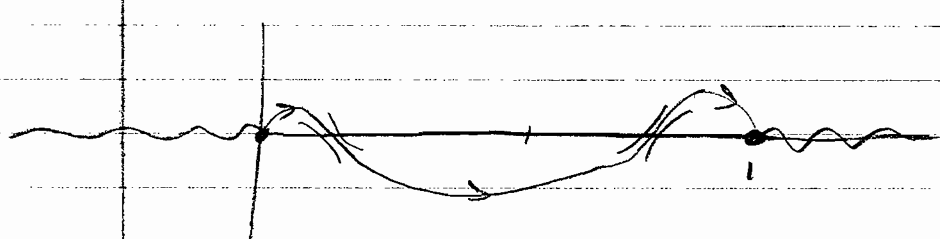
$$\boxed{\psi \sim \sqrt{\frac{\pi}{8\lambda}}\, t^{1/2} e^{2\lambda t} e^{-\lambda\pi} e^{-i2\lambda t^2} \qquad t \ll 1}$$

$t \gg 1$: saddle direction: $\pm i\Delta w^2 \sim -1$

For w_+, $\Delta w^2 \sim i$ $\quad \Delta w \sim e^{i\pi/4},\ e^{i5\pi/4}$

For w_-, $\Delta w^2 \sim -i$ $\quad \Delta w \sim e^{i3\pi/4},\ e^{i7\pi/4}$

$w_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{1}{t^2}}$ $\qquad 1-w_\pm = \frac{1}{2} \mp \frac{1}{2}\sqrt{1-\frac{1}{t^2}} = w_\mp$



$\phi(w_\pm) = i\lambda\left[\ln w_\pm - \ln(1-w_\pm) - 4w_\pm t^2\right]$

$\quad = i\lambda\left[\ln w_\pm - \ln w_\mp\right] - 4i\lambda\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{1}{t^2}}\right]t^2$

$\frac{w_\pm}{w_\mp} = \frac{1 \pm \sqrt{1-\frac{1}{t^2}}}{1 \mp \sqrt{1-\frac{1}{t^2}}} \approx \frac{1 \pm (1-\frac{1}{2t^2})}{1 \mp (1-\frac{1}{2t^2})}$ $\qquad$ upper sign: $4t^2$ $\quad$ lower sign: $\frac{1}{4t^2}$

$\phi(w_+) = i\lambda \ln(4t^2) - 4i\lambda t^2$

$\phi(w_-) = -i\lambda \ln(4t^2) - 2i\lambda$

$\phi''|_{w_\pm} \approx \pm 16 i\lambda t^4$

Saddle contribution: from w_-: $\displaystyle\int_{w_- - \infty e^{i\pi/4}}^{w_- + \infty e^{i\pi/4}} dw\, e^{\frac{1}{2}\phi''(w-w_-)^2} = 2\int_{w_-}^{w_- + \infty e^{-i\pi/4}} dw\, e^{\frac{1}{2}\phi''(w-w_-)^2}$

$u = e^{i\pi/2} 8\lambda t^4 (w-w_-)^2$ $\qquad (w-w_-) = \sqrt{\frac{u}{8\lambda t^4}}\, e^{-i\pi/4}$

$du = e^{i\pi/2} 16\lambda t^4 (w-w_-)\,dw$ $\qquad du = 2 \cdot e^{i\pi/4}\sqrt{8\lambda t^4}\sqrt{u}\,dw$

$\frac{2}{2e^{i\pi/4}\sqrt{8\lambda t^4}} \displaystyle\int_0^\infty du\, \frac{e^{-u}}{\sqrt{u}} = \frac{e^{-i\pi/4}\sqrt{\pi}}{t^2\sqrt{8\lambda}}$

$\to$ from w_-, get $e^{-i\lambda\ln(4t^2) - 2i\lambda} \cdot \frac{\sqrt{\pi}\, e^{-i\pi/4}}{t^2\sqrt{8\lambda}}$

From W_+: $\displaystyle 2\int_{w_+}^{w_+ + \infty e^{i\pi/4}} dw \quad e^{\frac{1}{2}\phi''(w-w_+)^2}$

$\phi''(w_+) = 16 i \lambda t^4 \quad = 16\, e^{i\pi/2} \lambda t^4$

$u = e^{-i\frac{\pi}{2}} 8 \lambda t^4 (w-w_+)^2 \qquad du = 2\cdot e^{-i\frac{\pi}{2}}\, 8\lambda t^4 (w-w_+)\, dw$

$(w-w_+) = e^{i\pi/4}\sqrt{\frac{u}{8\lambda t^4}} \qquad du = 2\, e^{-i\pi/4}\sqrt{8\lambda t^4}\sqrt{u}\, dw$

$$\rightarrow \frac{2\cdot}{2e^{-i\pi/4}\sqrt{8\lambda t^4}} \int_0^\infty du\, \frac{e^{-u}}{\sqrt{u}} = \frac{\sqrt{\pi}\, e^{i\pi/4}}{\sqrt{8\lambda}\, t^2}$$

$$\rightarrow \text{get} \quad e^{i\lambda \ln(4t^2)} e^{-4i\lambda t^2} \frac{\sqrt{\pi}\, e^{i\pi/4}}{\sqrt{8\lambda}\, t^2}$$

$$\psi \sim \sqrt{\frac{\pi}{8\lambda}} \left[e^{i\lambda \ln(4t^2)} e^{i\pi/4} e^{-4i\lambda t^2} + e^{-i\lambda \ln(4t^2)} e^{-i\pi/4} e^{-2i\lambda} \right]$$