2004 Part I Q7 — Nonneutrals

a. $H = mc^2\gamma - e\phi \qquad \gamma = \left(1+\frac{p^2}{m^2c^2}\right)^{1/2}$

$$\frac{dH}{dt} = mc^2\frac{d\gamma}{dt} - e\frac{d\vec{r}}{dt}\cdot\nabla\phi = mc^2\frac{d\gamma}{dt} + e\vec{v}\cdot\vec{E}$$

$$\frac{d\gamma}{dt} = \frac{1}{\gamma}\frac{\vec{p}}{m^2c^2}\cdot\frac{d\vec{p}}{dt} = \frac{\vec{v}}{mc^2}\cdot\frac{d\vec{p}}{dt}$$

$$\frac{dH}{dt} = mc^2\cdot\frac{\vec{v}}{mc^2}\cdot\frac{d\vec{p}}{dt} + e\vec{v}\cdot\vec{E}$$

$$= \vec{v}\cdot -e\left[E_r\hat{e}_r + \frac{1}{c}\vec{v}\times(B_\theta\hat{e}_\theta + B_z\hat{e}_z)\right] + e\vec{v}\cdot\vec{E}$$

$$\frac{dH}{dt} = -e\vec{v}\cdot E_r\hat{r} + e\vec{v}\cdot E_r\hat{r} = 0$$

$$\frac{dP_\theta}{dt} = \frac{d}{dt}\left\{r\left[p_\theta - \frac{e}{c}A_\theta(r)\right]\right\}$$

$$\frac{d}{dt}(rA_\theta) = \frac{d\vec{r}}{dt}\cdot\nabla(rA_\theta) = v_r\frac{d}{dr}(rA_\theta) = rv_r B_z(r)$$

$$\frac{d}{dt}(rp_\theta) = m\frac{d}{dt}(r\gamma v_\theta) = m\frac{d}{dt}\left(r^2\gamma\frac{d\theta}{dt}\right)$$

$$= m\left[2r\frac{dr}{dt}\gamma\frac{d\theta}{dt} + r^2\frac{d\gamma}{dt}\frac{d\theta}{dt} + r^2\gamma\frac{d^2\theta}{dt^2}\right]$$

$$= r\left\{\gamma m\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] + mr\frac{d\gamma}{dt}\frac{d\theta}{dt}\right\} = r\cdot\left(\frac{d\vec{p}}{dt}\right)_\theta = r\cdot\frac{e}{c}v_r B_z$$

$$\frac{dP_\theta}{dt} = r\frac{e}{c}v_r B_z - \frac{e}{c}rv_r B_z = 0$$

$$\frac{dP_z}{dt} = \frac{d}{dt}\left(p_z - \frac{e}{c}A_z\right) = \frac{dp_z}{dt} - \frac{e}{c}v_r\frac{dA_z}{dr} = \frac{dp_z}{dt} + \frac{e}{c}v_r B_\theta$$

$$\frac{dp_z}{dt} = -\frac{e}{c}v_r B_\theta$$

$$\Rightarrow \frac{dP_z}{dt} = 0$$

b. $n_b(r) = \begin{cases} \hat{n}_b = \text{const} & 0 \le r < r_b \\ 0 & r > r_b \end{cases}$

$J_{zb}(r) = \begin{cases} -eV_b\hat{n}_b = \text{const} & 0 \le r < r_b \\ 0 & r > r_b \end{cases}$

$\nabla^2 \phi = -4\pi q n = 4\pi e \hat{n}_b \qquad r < r_b$

$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\phi(r) = 4\pi e \hat{n}_b \qquad r\frac{d}{dr}\phi = 2\pi e \hat{n}_b r^2$

$\frac{d\phi}{dr} = 2\pi e \hat{n}_b r \longrightarrow \phi = \pi e \hat{n}_b r^2 \qquad$ choosing $\phi(0)=0$

$= \frac{1}{4}\frac{m}{e}\omega_{pb}^2 r^2 \qquad r<r_b$

$H = mc^2\gamma - \frac{1}{4} m\omega_{pb}^2 r^2$

$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} \implies \nabla \times (\nabla \times \vec{A}) = \frac{4\pi}{c}\vec{J}$

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$

- Work in the Coulomb gauge; $\nabla \cdot \vec{A} = 0$

$\Rightarrow \nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{J} \qquad$ Therefore only A_z is nonzero

$\nabla^2 A_z = -\frac{4\pi}{c} J_z = \frac{4\pi e V_b \hat{n}_b}{c} \qquad r < r_b$

$\frac{r\, dA_z}{dr} = \frac{2\pi e V_b \hat{n}_b}{c} r^2 \longrightarrow A_z = \frac{\pi e V_b \hat{n}_b}{c} r^2 \qquad$ choosing $A_z(0)=0$

$A_z = \frac{V_b}{4c}\frac{m}{e}\omega_{pb}^2 r^2 \qquad r<r_b$

$P_z = p_z - \frac{1}{4} m V_b \frac{\omega_{pb}^2 r^2}{c^2} \qquad r<r_b$