2013 II Q2 JH

Take a deep breath. It's not that bad.

Maxwell-Vlasov System, $\vec{k} = k\hat{z}$

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f = -\frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}}$$

$f = f_0 + f_1$ $\quad \vec{E}, \vec{B}$ both first order

$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} \implies \vec{B}_1 = +\frac{k}{\omega}\hat{z}\times\vec{E}$$

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\nabla f_1 = +\frac{e}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\frac{\partial f_0}{\partial \vec{v}}$$

Fourier Transform: $\quad \to k_z v_z \quad k$ in $\hat{z}$

$$(-i\omega + i\vec{k}\cdot\vec{v})f_1 = \frac{e}{m}\left(\vec{E} - \frac{k}{\omega}\vec{v}\times(\hat{z}\times\vec{E})\right)\cdot\frac{\partial f_0}{\partial \vec{v}}$$

Now we want $\overleftrightarrow{\varepsilon} = \overleftrightarrow{1} + \frac{4\pi i\overleftrightarrow{\sigma}}{\omega}$ $\quad \vec{j} = \overleftrightarrow{\sigma}\cdot\vec{E}$ and $\vec{j} = -en\int \vec{v} f_1 d^3v$

$$f_1 = -\frac{e}{m}\frac{\left(\vec{E} - k/\omega\,\vec{v}\times(\hat{z}\times\vec{E})\right)}{i(\omega - kv_z)}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

$$\hat{z}\times\vec{E} = \begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix} \qquad \vec{v}\times(\hat{z}\times\vec{E}) = \vec{E}\cdot\vec{v}\,\hat{z} - v_z\vec{E}$$

$$f_1 = -\frac{e}{m}\frac{1}{i\omega}\frac{\vec{E}(\omega - kv_z) + \vec{E}\cdot\vec{v}\,\hat{z}k}{\omega - kv_z}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

$$j = \frac{e^2}{im\omega}\int\left(\vec{E} + \frac{k\vec{E}\cdot\vec{v}\,\hat{z}}{\omega - kv_z}\right)\vec{v}\cdot\frac{\partial f_0}{\partial \vec{v}}$$

Now, $\int v_j \frac{\partial f_0}{\partial v_i} d^3v = \cancel{[v_j f_0]}^{\,0} - \int \frac{\partial v_j}{\partial v_i} f_0\, d^3v$

$= -\delta_{ij}$ as $\int f_0\, d^3v = 1$

$$\frac{\vec{E}\cdot\vec{v}\,\vec{v}\,\hat{z}}{\omega - kv_z}\cdot\frac{\partial f_0}{\partial \vec{v}} = \frac{\vec{v}\vec{v}\vec{E}}{\omega - kv_z}\frac{\partial f_0}{\partial v_z}$$

$$= \underbrace{\frac{e^2}{im\omega}\left(-\mathbb{I} + \int\frac{\vec{v}\vec{v}k}{\omega - kv_z}\frac{\partial f_0}{\partial v_z}\right)}\vec{E}$$

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{\mathbb{I}} + \frac{4\pi i\overleftrightarrow{\sigma}}{\omega} = \overleftrightarrow{\mathbb{I}}\left(1 - \frac{\omega_p^2}{\omega^2}\right) + \frac{\omega_p^2}{\omega^2}\int\frac{kv_iv_j}{\omega - kv_z}\frac{\partial f_0}{\partial v_z}$$ as required.

b) $$f_0(\vec{v}) = \frac{1}{\pi^{3/2}W_\perp^2 W_\parallel}\exp\left(-\frac{v_x^2}{W_\perp^2} - \frac{v_y^2}{W_\perp^2} - \frac{v_z^2}{W_\parallel^2}\right)$$

ε_{ij} diagonal: prove $\int \frac{v_iv_j}{\omega - kv_z}\frac{\partial f_0}{\partial v_z} d^3v$ diagonal.

$$\frac{\partial f_0}{\partial v_z} = -\frac{2v_z}{W_\parallel} f_0$$

$\int v_i f_0\, d^3v = 0$ by symmetry

$\int v_i^2 f_0\, d^3v \neq 0$

Hence $\int \frac{v_iv_jv_z \frac{\partial f_0}{\partial v_z}}{\omega - kv_z} = 0$ if $v_i \neq v_j$

Find ε_{xx} explicitly:

$$\frac{\omega_p^2}{\omega^2}\int \frac{k V_x^2}{\omega - kV_z} \cdot \frac{-2V_z}{W_\parallel} \, \frac{1}{\pi^{3/2} W_\perp^2 W_\parallel} \exp\left(-\frac{V_x^2}{W_\perp^2} - \frac{V_y^2}{W_\perp^2} - \frac{V_z^2}{W_\parallel^2}\right) dV_x \, dV_y \, dV_z$$

dV_y : $\quad y = \frac{V_y}{W_\perp} \quad dV_y = W_\perp \, dy \quad W_\perp \int e^{-y^2} dy = \sqrt{\pi} \, W_\perp$

dV_x : $\quad x = \frac{V_x}{W_\perp} \quad dV_x = W_\perp \, dx \quad W_\perp^3 \int x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} W_\perp^3$

$$\Rightarrow -\frac{\omega_p^2}{\omega^2} \frac{W_\perp^2}{W_\parallel} \frac{1}{\sqrt{\pi}} \int \frac{kV_z/W_\parallel}{\omega - kV_z} \exp\left(-\frac{V_z^2}{W_\parallel^2}\right) dV_z$$

Let $t = \frac{V_z}{W_\parallel} \quad \zeta = \frac{\omega}{kW_\parallel} \quad \Rightarrow -\frac{\omega_p^2}{\omega^2} \frac{W_\perp^2}{W_\parallel^2} \frac{1}{\sqrt{\pi}} \int \frac{t e^{-t^2}}{\zeta - t} dt$

$$\int \frac{t e^{-t^2}}{t-\zeta} dt = \int \frac{t-\zeta+\zeta}{t-\zeta} e^{-t^2} dt = \sqrt{\pi} + \int \frac{\zeta e^{-t^2}}{t-\zeta} dt$$

$\Rightarrow \frac{\omega_p^2}{\omega^2} \frac{T_\perp}{T_\parallel} \left[1 + \zeta Z(\zeta)\right]$ Using defn of Z.

Get dispersion w/ $n^2 - \varepsilon_{xx} = 0$
$n^2 = \frac{k^2c^2}{\omega^2}$

c) Hot, $\zeta \ll 1 \quad Z(\zeta) = -2\zeta$

$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2 \frac{T_\perp}{T_\parallel}\left[1 - 2\zeta^2\right] = 0$ Unstable, $\omega^2 < 0$
$\zeta^2 < 0$ so $1 - 2\zeta^2 > 1$

$k^2c^2 < \omega_p^2 \left[\frac{T_\perp}{T_\parallel}\left[1-2\zeta^2\right] - 1\right]$ $\quad \frac{T_\perp}{T_\parallel}\left[1-2\zeta^2\right] > 1$

Always some ~~unstable~~ k which is unstable

d) $Z(\zeta \gg 1) \sim -\frac{1}{\zeta} - \frac{1}{2\zeta^3}$

$$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2 \frac{T_\perp}{T_\parallel}\left[\cancel{1-1} - \frac{1}{2\zeta^2}\right] = 0$$

$$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2 \frac{T_\perp}{T_\parallel} \cdot -\frac{k^2 w_\parallel^2}{2\omega^2} = 0$$

$$\omega^4 - \left(k^2c^2 + \omega_p^2\right)\omega^2 - \frac{\omega_p^2 T_\perp k^2}{m} = 0$$

$$\omega^2 = \frac{k^2c^2 + \omega_p^2}{2} \pm \frac{1}{2}\sqrt{\underbrace{(k^2c^2+\omega_p^2)^2}_{>0} + \underbrace{\frac{4\omega_p^2 T_\perp k^2}{m}}_{>0}}$$

Hence $-$ sign gives $\omega^2 < 0$ as $(\;)^{1/2} > \frac{k^2c^2+\omega_p^2}{2}$

Assume $k^2c^2 + \omega_p^2 \gg \frac{4\omega_p^2 T_\perp k^2}{m}$

Unstable: $\omega^2 = \frac{k^2c^2+\omega_p^2}{2} - \frac{k^2c^2+\omega_p^2}{2}\left(1 + \frac{4\omega_p^2 T_\perp k^2}{m(k^2c^2+\omega_p^2)^2}\right)^{1/2}$

$$\simeq -\frac{\omega_p^2 T_\perp k^2}{m(k^2c^2+\omega_p^2)^2} \simeq -\frac{T_\perp k^2}{m} \quad \text{if } k^2c^2 \ll \omega_p^2$$

Cold plasma: $\frac{\omega}{k w_\parallel} \gg 1 \Rightarrow \frac{T_\perp}{m} \gg \frac{T_\parallel}{m}$