Jan 2004 #2 (CM)

mass m $\quad F \propto r^{-n}$, $\quad U(r) = \frac{-k}{r^{n-1}}$

$r(\theta) = 2a\cos\theta$

a. Derivation of orbit equation to find the force law:

Lagrangian eqn of motion for r, then substitute $u = \frac{1}{r}$:

$$L = T - U = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - U(r)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \qquad m(\ddot{r} - r\dot{\theta}^2) = \frac{-\partial U(r)}{\partial r} = F(r)$$

$$u = \frac{1}{r} \qquad \frac{du}{d\theta} = \frac{dr}{d\theta}\frac{du}{dr} = \frac{du}{dr}\frac{\dot{r}}{\dot{\theta}} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \qquad \dot{\theta} = \frac{\ell}{mr^2}$$

$$\frac{du}{d\theta} = -\frac{m}{\ell}\dot{r}$$

$$\frac{d^2u}{d\theta^2} = \frac{-m}{\ell}\frac{d}{d\theta}\dot{r} = \frac{-m}{\ell}\frac{dt}{d\theta}\frac{d}{dt}\dot{r} = \frac{-m}{\ell}\frac{1}{\dot{\theta}}\ddot{r} = \frac{-m^2r^2\ddot{r}}{\ell^2}$$

$$\rightarrow \ddot{r} = -\frac{\ell^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

$$r\dot{\theta}^2 = \frac{\ell^2}{m^2r^3} = \frac{\ell^2u^3}{m^2}$$

$$\Rightarrow -\frac{\ell^2}{m}u^2\frac{d^2u}{d\theta^2} - \frac{\ell^2}{m}u^3 = F\left(\frac{1}{u}\right)$$

$$\Rightarrow \frac{\ell^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = -F(r)$$

$$r = 2a\cos\theta \qquad \frac{1}{r} = \frac{1}{2a}\sec\theta \qquad \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{1}{2a}\sec\theta\tan\theta$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{2a}(\sec\theta\tan^2\theta + \sec^3\theta) = \frac{1}{2a}\sec\theta(\sec^2\theta + \tan^2\theta)$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{2a}\sec\theta(\sec^2\theta + \tan^2\theta + 1) = \frac{1}{2a}(2\sec^2\theta)\sec\theta = \frac{\sec^3\theta}{a}$$

$$\sec^3\theta = \frac{8a^3}{r^3}$$

$$F(r) = \frac{-\ell^2}{mr^2}\cdot\frac{8a^2}{r^3} = \frac{-8a^2\ell^2}{mr^5}$$

$$n = 5$$

b. $-\frac{dU}{dr} = F(r) \qquad U(r) = \frac{-2a^2\ell^2}{mr^4} \qquad k = \frac{2a^2\ell^2}{m} \rightarrow \ell = \sqrt{\frac{mk}{2a^2}}$

c. $E = T + U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{2a^2\ell^2}{mr^4}$

$r = 2a\cos\theta \qquad \dot{r} = -2a\sin\theta\dot{\theta} \qquad \dot{r}^2 = 4a^2\sin^2\theta\,\dot{\theta}^2 = \frac{4a^2\ell^2\sin^2\theta}{m^2r^4}$

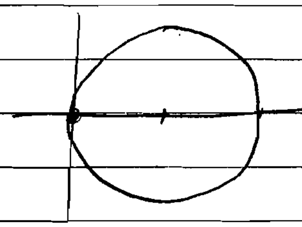
$r^2\dot{\theta}^2 = \frac{\ell^2}{mr^2} = \frac{\ell^2}{4a^2m\cos^2\theta}$

$$E = \frac{2a^2\ell^2\sin^2\theta}{m\cdot 16a^4\cos^4\theta} + \frac{\ell^2}{8a^2m\cos^2\theta} - \frac{2a^2\ell^2}{m\cdot 16a^4\cos^4\theta}$$

$$= \frac{\ell^2}{8a^2m}\left(\frac{\sin^2\theta + \cos^2\theta}{\cos^4\theta} - \frac{1}{\cos^4\theta}\right) = 0$$

$E = 0$

d. $r = 2a\cos\theta$



circular orbit, radius a

orbit starts at $r=0$ at $\theta = \frac{\pi}{2}$, returns at $\theta = \frac{3\pi}{2}$

$\frac{d\theta}{dt} = \frac{\ell}{mr^2} \qquad \frac{mr^2}{\ell}d\theta = dt$

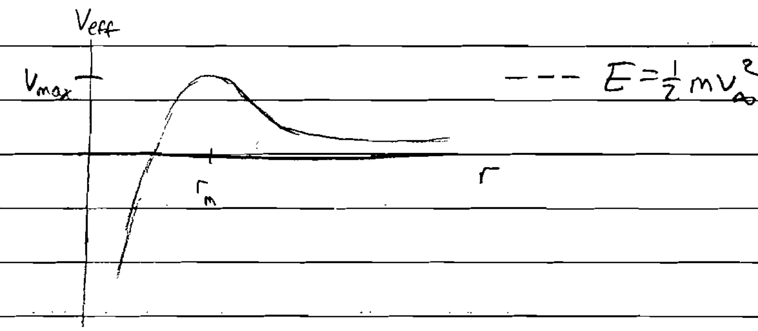
$$T = \frac{m}{\ell}\int_{\frac{\pi}{2}}^{2\pi/2} d\theta\, r^2 = \frac{4a^2m}{\ell}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^2\theta\, d\theta = \frac{2a^2m}{\ell}\int_{\frac{\pi}{2}}^{3\pi/2}(1+\cos 2\theta)\,d\theta$$

$$T = \frac{2a^2m}{\ell}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/2}^{3\pi/2} = \frac{2a^2m}{\ell}\left[\pi\right]$$

$$T = 2\pi\frac{a^2m}{\ell}$$

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{r^4} = \tfrac{1}{2}mv_\infty^2 = \tfrac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{k}{r^4}$$

$$V_{eff} = \frac{\ell^2}{2mr^2} - \frac{k}{r^4}$$



To reach $r=0$, the particle needs an initial energy $E = \tfrac{1}{2}mv_\infty^2 > V_{max}$

Initial $\ell = mbv_\infty$

V_{max}: $\dfrac{dV_{eff}}{dr} = 0$: $\quad -\dfrac{\ell^2}{mr^3} + \dfrac{4k}{r^5} = 0 \qquad \dfrac{\ell^2 r_m^2}{m} = 4k$

$$r_m^2 = \frac{4mk}{\ell^2}$$

$$V_{max} = V_{eff}(r_m) = \frac{\ell^2}{2m \cdot 4mk}\ell^2 - \frac{k\,\ell^4}{16m^2k^2}$$

$$V_{max} = \frac{\ell^4}{8m^2k} - \frac{\ell^4}{16m^2k} = \frac{\ell^4}{16m^2k} = \frac{m^4b^4v_\infty^4}{16m^2k}$$

Capture: $E > V_{max}$: $\quad \tfrac{1}{2}mv_\infty^2 > \dfrac{m^2b^4v_\infty^4}{16k}$

$$b^4 < \frac{8k}{mv_\infty^2}$$

maximum capture impact parameter is the equal sign:

$$b_{max}^2 = \sqrt{\frac{8k}{mv_\infty^2}}$$

$$\sigma = A = \pi b_{max}^2$$

$$\sigma = \pi\sqrt{\frac{8k}{mv_\infty^2}}$$