Jan 2001 #2 (EM)

Relativistic cyclotron motion:

$$\frac{d\vec{p}}{dt} = \frac{q\vec{v}\times\vec{B}}{c} \qquad \frac{d(m\gamma\vec{v})}{dt} = \frac{q\,\vec{v}\times\vec{B}}{c}$$

For pure circular motion at constant speed, γ = constant

$\Rightarrow$ same solution as non-relativistic cyclotron motion with $m \Rightarrow m\gamma$

$$\Omega = \frac{qB}{m\gamma c} \qquad |\Omega| = \frac{eB}{m\gamma c}$$

$$R = \frac{v}{|\Omega|} = \frac{c}{e}\,\frac{m\gamma v}{B} = \frac{c}{e}\,\frac{p}{B} \qquad p = |\vec{p}| \text{ magnitude of } \vec{p}$$

The problem is asking what the condition on B is to make R a constant

$$\Rightarrow \frac{dR}{dt} = 0 \qquad \frac{1}{B}\frac{dp}{dt} - \frac{p}{B^2}\frac{dB}{dt} = 0 \qquad \frac{B}{p}\frac{dp}{dt} = \frac{dB}{dt}$$

$\frac{dp}{dt}$? $\vec{p}$ changes magnitude because of an inductive electric field, not the magnetic field directly:

$$\frac{dp}{dt} = -eE_\phi$$

E_ϕ is determined by Faraday's law:

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \int \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\iint \vec{B}\cdot d\vec{A}$$

$$2\pi R E = \frac{-\pi R^2}{c}\frac{dB_{z\text{-ave}}}{dt}$$

$$E_\phi = \frac{-R}{2c}\frac{dB_{z\text{-ave}}}{dt}$$

$$\Rightarrow \frac{dp}{dt} = \frac{eR}{2c}\frac{dB_{z\text{-ave}}}{dt}$$

$$\Rightarrow \frac{eR}{2c}\,\frac{B}{p}\,\frac{dB_{z\text{-ave}}}{dt} = \frac{dB}{dt} \qquad R = \frac{c}{e}\frac{p}{B}$$

$$\Rightarrow \frac{c}{e}\frac{p}{B}\cdot\frac{e}{2c}\,\frac{B}{p}\,\frac{dB_{z\text{-ave}}}{dt} = \frac{dB}{dt}$$

$$\Rightarrow \frac{1}{2}\frac{dB_{z\text{-ave}}}{dt} = \frac{dB}{dt}$$

$$\frac{d}{dt}\left(B - \frac{B_{z\text{-ave}}}{2}\right) = 0 \qquad B = \frac{B_{z\text{-ave}}}{2} + \text{const}$$

maximum energy: power in = power radiated

$$P_{rad} = \frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right]$$

For relativistic circular motion, $\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\Omega|\vec{p}| >> \frac{1}{c}\frac{dE}{d\tau}$ $\qquad |p| = m\gamma v$

$$P_{rad} = \frac{2}{3}\frac{e^2}{m^2c^3}\cdot\gamma^2\Omega^2\cdot m^2\gamma^2v^2 = \frac{2}{3}\frac{e^2\Omega^2v^2\gamma^4}{c^3}$$

$$\Omega = \frac{v}{R} \quad \text{or} \quad \Omega = \frac{eB}{m\gamma c}$$

ultra-relativistic: $v \cong c$; $\Rightarrow$ $R \cong \frac{E}{eB}$ $\quad (E >> mc^2)$ $\quad$ (energy $E = \gamma mc^2$)

$$\Rightarrow P_{rad} = \frac{2}{3}\frac{e^2}{m^4c^7}\frac{E^4}{R^2} = \frac{2}{3}\frac{e^2}{m^4c^7}e^2B^2E^2 \quad \text{(equivalent forms)}$$

$$P_{in} = -e\vec{E}\cdot\vec{v} = \frac{eR}{2}\frac{dB_{z\text{-ave}}}{dt}$$

maximum E when $P_{rad} = P_{in}$

$$\frac{2}{3}\frac{e^2}{m^4c^7}\frac{E^4}{R^2} = \frac{eR}{2}\frac{dB_{z\text{-ave}}}{dt}$$

$$\boxed{E^4 = \frac{3}{4}\frac{m^4c^7}{e}R^3\frac{dB_{z\text{-ave}}}{dt} \quad \text{or} \quad E = \frac{3}{4}\frac{m^4c^7}{e^4}\frac{1}{B^3}\frac{dB_{z\text{-ave}}}{dt}}$$