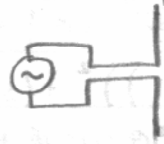
Seth Dorfman Prelims December 24, 2005

January 2000 EM

3) For one dipole:



$$I(z,t) = I_o e^{i(kz-\omega t)} \qquad \frac{\omega}{k} = c$$

$$J(z,t) = \frac{I_o}{d} e^{i(kz-\omega t)}$$

$$\vec{\nabla}\cdot\vec{J} = ik\frac{I_o}{d} e^{i(kz-\omega t)} = -\frac{\partial\rho}{\partial t}$$

$$\Rightarrow \rho(z,t) = \frac{k}{\omega}\frac{I_o}{d} e^{i(kz-\omega t)}$$

$$\rho(z,t) = \frac{I_o}{cd} e^{i(kz-\omega t)}$$

$$\vec{d} = \int \rho\vec{r}$$

$$= \int_{-d/2}^{d/2} z\frac{I_o}{cd} e^{i(kz-\omega t)}\,dz\,\hat{z}$$

$$= \frac{I_o}{cd} e^{-i\omega t}\int_{-d/2}^{d/2} z e^{ikz}\,dz\,\hat{z} \qquad \text{Let } u = z \quad dv = e^{ikz}dz \quad du = dz \quad v = -\frac{i}{k}e^{ikz}$$

$$= \frac{I_o}{cd} e^{-i\omega t}\left[-\frac{i}{k} z e^{ikz}\Big|_{-d/2}^{d/2} + \frac{i}{k}\int_{-d/2}^{d/2} e^{ikz}\,dz\right]\hat{z}$$

$$= \frac{I_o}{cd} e^{-i\omega t}\left[-\frac{i}{k}\frac{d}{2}\left(e^{ikd/2} + e^{-ikd/2}\right) + \frac{i}{k}\cdot -\frac{i}{k} e^{ikz}\Big|_{-d/2}^{d/2}\right]\hat{z}$$

$$= \frac{I_o}{cd} e^{-i\omega t}\left[i\frac{d}{k}\cos\left(\frac{kd}{2}\right) - \frac{1}{k^2}\left(e^{ikd/2} - e^{-ikd/2}\right)\right]\hat{z}$$

$$= \frac{I_o}{cd} e^{-i\omega t}\left[i\frac{d}{k}\cos\left(\frac{kd}{2}\right) - \frac{2i}{k^2}\sin\left(\frac{kd}{2}\right)\right]\hat{z}$$

$$\vec{d} = i\frac{I_o}{ck} e^{-i\omega t}\left[\cos\left(\frac{kd}{2}\right) - \frac{2}{kd}\sin\left(\frac{kd}{2}\right)\right]\hat{z}$$

Since $\frac{d}{2} << \lambda$, $\frac{kd}{2} << 1$

$$\therefore \vec{d} \approx i\frac{I_o}{ck} e^{-i\omega t}\left[1 - \frac{1}{2}\left(\frac{kd}{2}\right)^2 - \frac{2}{kd}\cdot\frac{kd}{2} + \frac{2}{kd}\cdot\frac{1}{6}\left(\frac{kd}{2}\right)^3\right]\hat{z}$$

$$\vec{d} \approx -i\frac{I_o}{ck} e^{-i\omega t}\frac{1}{3}\left(\frac{kd}{2}\right)^2\hat{z}$$

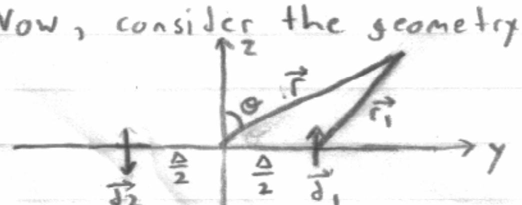
$$\vec{d} \approx -i\frac{I_o}{12c} kd^2 e^{-i\omega t}\hat{z}$$

$$\vec{d} \approx -i\frac{I_o}{12c^2}\omega d^2 e^{-i\omega t}\hat{z}$$

$$\dot{\vec{d}} = -\frac{I_o}{12c^2}\omega^2 d^2 e^{-i\omega t}\hat{z}$$

$$\ddot{\vec{d}} = i\frac{I_o}{12c^2}\omega^3 d^2 e^{-i\omega t}\hat{z}$$

Now, consider the geometry of two dipoles:



$$\vec{r} = \frac{\Delta}{2}\hat{y} + \vec{r}_1$$

$$\hat{r} = \frac{\Delta}{2r}\hat{y} + \frac{\vec{r}_1}{r}$$

$$|\vec{r}_1| = \left|\vec{r} - \frac{\Delta}{2}\hat{y}\right|$$

$$= \sqrt{r^2 - r\Delta\sin\theta\sin\phi + \frac{\Delta^2}{4}}$$

$$= r\sqrt{1 - \frac{\Delta\sin\theta\sin\phi}{r} + \left(\frac{\Delta}{2r}\right)^2}$$

$$\approx r\left(1 - \frac{\Delta\sin\theta\sin\phi}{2r} + \frac{1}{2}\left(\frac{\Delta}{2r}\right)^2\right)$$

$$\approx r - \frac{\Delta}{2}\sin\theta\sin\phi \quad \text{(large } r\text{)}$$

$$|\vec{r}_1| = r - \frac{\Delta}{2}\sin\theta\sin\phi$$

$$|\vec{r}_2| = r + \frac{\Delta}{2}\sin\theta\sin\phi$$

$$\ddot{\vec{d}}_1\left(t-\frac{r_1}{c}\right) = i\,\frac{I_0}{12c^2}\,w^3 d^2\, e^{-i\left[w\left(t-\frac{1}{c}\left(r-\frac{\Delta}{2}\sin\theta\sin\phi\right)\right)-\alpha/2\right]}\ \hat{z}$$

$$\ddot{\vec{d}}_2\left(t-\frac{r_2}{c}\right) = i\,\frac{I_0}{12c^2}\,w^3 d^2\, e^{-i\left[w\left(t-\frac{1}{c}\left(r+\frac{\Delta}{2}\sin\theta\sin\phi\right)\right)+\alpha/2\right]}\ \hat{z}$$

$\alpha \equiv$ phase difference between dipoles

$$\ddot{\vec{d}}_1+\ddot{\vec{d}}_2 = i\,\frac{I_0}{12c^2}\,w^3 d^2\, e^{-iw\left(t-\frac{r}{c}\right)}\left[e^{i\left[\frac{w\Delta}{2c}\sin\theta\sin\phi+\frac{\alpha}{2}\right]}\ \hat{z} + e^{-i\left[\frac{w\Delta}{2c}\sin\theta\sin\phi+\frac{\alpha}{2}\right]}\ \hat{z}\right.$$

$$\ddot{\vec{d}}_1+\ddot{\vec{d}}_2 = i\,\frac{I_0}{6c^2}\,w^3 d^2\cos\left(\frac{w\Delta}{2c}\sin\theta\sin\phi+\frac{\alpha}{2}\right)e^{-iw\left(t-\frac{r}{c}\right)}\ \hat{z}$$

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\left(\ddot{\vec{d}}_1+\ddot{\vec{d}}_2\right)\times\hat{r}\right)^2$$

$$= \frac{1}{4\pi c^3}\left(\frac{I_0}{6c^2}w^3 d^2\right)^2\cos^2\left(\frac{w\Delta}{2c}\sin\theta\sin\phi+\frac{\alpha}{2}\right)\cos^2\left(w\left(t-\frac{r}{c}\right)\right)\left(\hat{z}\times\hat{r}\right)^2$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{4\pi c^3}\left(\frac{I_0}{6}wK^2 d^2\right)^2\sin^2\theta\cos^2\left(\frac{w\Delta}{2c}\sin\theta\sin\phi+\frac{\alpha}{2}\right)\cdot\frac{1}{2}$$

Plugging in $\Delta = \frac{\lambda}{2}$, $\alpha = \pi$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{8\pi c^3}\left(\frac{I_0}{6}\,w\,\frac{4\pi^2}{\lambda^2}d^2\right)^2\sin^2\theta\sin^2\left(\frac{w\lambda}{4c}\sin\theta\sin\phi\right)$$

$$= \frac{1}{2\pi c^3}\left(\frac{I_0}{3}\,w\,\pi^2\left(\frac{d}{\lambda}\right)^2\right)^2\sin^2\theta\sin^2\left(\frac{w}{K}\frac{\pi}{2c}\sin\theta\sin\phi\right)$$

$$= \frac{1}{18\pi c^3}\,I_0^2 w^2\pi^4\left(\frac{d}{\lambda}\right)^4\sin^2\theta\sin^2\left(\frac{\pi}{2}\sin\theta\sin\phi\right)$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{I_0^2 w^2\pi^3}{18c^3}\left(\frac{d}{\lambda}\right)^4\sin^2\theta\sin^2\left(\frac{\pi}{2}\sin\theta\sin\phi\right)$$