General Examination May 9, 2011 Part II

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1B Magnetohydrodynamics [20 Points]

a) [5 Points] Write down the equations of motion for the electron and the ion fluids, assuming that their pressures are scalar, and that the ions are singly ionized.

Continuity

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \vec{v}_s) = 0$$

Momentum

$$\frac{\partial}{\partial t}(\rho_s \vec{v}_s) + \nabla \cdot (\rho_s \vec{v}_s \vec{v}_s) + \nabla P_S - e_s n_s \left(\vec{E} + \frac{1}{c} \vec{v}_s \times \vec{B} \right) - R_s = 0$$

Closure

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma} \right) = 0$$

b) [5 Points] Ignoring the electron mass, derive the one fluid MHD equation of motion and the generalized Ohm's law from the force-balance equations in part (a).

Sum the ion and electron momentum equations

$$\frac{\partial}{\partial t}(\rho_i \vec{v}_i + \rho_e \vec{v}_e) + \nabla \cdot (\rho_i \vec{v}_i \vec{v}_i + \rho_e \vec{v}_e \vec{v}_e) + \nabla (P_e + P_i) - e(n_i - n_e)\vec{E} - \frac{e}{c}(n_i \vec{v}_i - n_e \vec{v}_e) \times \vec{B} - R_i - R_e = 0$$

Using the standard mass ratio expansion

$$\rho = m_i n_i + m_e n_e \approx m_i n$$

$$\vec{u} = \frac{m_i n_i \vec{v}_i + m_e n_e \vec{v}_e}{\rho} \approx \vec{v}_i$$

And

$$J = e(n_i \vec{v}_i - n_e \vec{v}_e)$$

We can rewrite this as

$$\frac{d}{dt}(\rho \vec{u}) + \nabla P - \frac{1}{c} J \times \vec{B} = 0$$

Now, from just the electron momentum equation

$$\frac{d}{dt}(\rho_e \vec{v}_e) + \nabla P_e + e n_e \left(\vec{E} + \frac{1}{c} \vec{v}_e \times \vec{B} \right) - R_e = 0$$

With some foresight, we substitute $R_e = e n_e \eta J$, and $e n_e \vec{v}_e = -J + e n_i \vec{v}_i = -J + e n \vec{u}$ to get the standard form of Ohm's law.

$$E + \frac{1}{c} \vec{u} \times \vec{B} - \eta J = \frac{\vec{J} \times \vec{B}}{c e n_e} - \frac{\nabla P_e}{e n_e} - \frac{1}{e n_e} \frac{d}{dt}(\rho_e \vec{v}_e)$$

c) [5 Points] What are the two conditions under which the generalized Ohm's law obtained in Part (b) can be reduced to the regular form of Ohm's law used in MHD?

The terms on the right hand side can be compared to the $\vec{u} \times \vec{B}$ term

$$\frac{\vec{J} \times \vec{B}}{cen_e} \frac{c}{\vec{u} \times \vec{B}} \sim \frac{ckB}{4\pi env_A} \sim \frac{ckB\omega_{pi}}{4\pi enc\Omega_i} \sim \frac{c/\omega_{pi}}{L} \ll 1$$

$$\frac{\nabla P_e}{en_e} \frac{c}{\vec{u} \times \vec{B}} \sim \beta \frac{ckB}{4\pi env_A} \sim \beta \frac{ckB\omega_{pi}}{4\pi enc\Omega_i} \sim \beta \frac{c/\omega_{pi}}{L} \ll 1$$

$$\frac{\nabla \cdot (\rho_e \vec{v}_e \vec{v}_e)}{en_e} \frac{c}{\vec{u} \times \vec{B}} \sim \frac{ckm_e v_A}{eB} \sim \frac{m_e ck}{m_i \omega_{pi}} \sim \frac{m_e}{m_i} \frac{c/\omega_{pi}}{L} \ll 1$$

$$\frac{\partial enm_e \vec{v}_e}{en\partial t} \frac{c}{\vec{u} \times \vec{B}} \sim \frac{\omega}{\Omega_e} \ll 1$$

d) [5 Points] Are these conditions typically satisfied in fusion plasmas? Can these two conditions be combined into a single condition?

Tokamak fusion experiments typically do not satisfy the MHD condition $\lambda_{mfp} \ll L$. The theory is still applicable, however, since $\rho_i \ll L$ and there are periodic boundary conditions.

Essentially we found two conditions (the second two terms in part (c) are automatically satisfied by the first). Both come down to the same physical property: our characteristic MHD time is much greater then the cyclotron frequency.

$$\frac{c/\omega_{pi}}{L} \sim \frac{v_A}{L} \frac{1}{\Omega_i} \sim \frac{\tau_{ci}}{\tau_A} \ll 1$$

$$\frac{\omega}{\Omega_e} \sim \frac{\tau_{ce}}{\tau_A} \ll 1$$

Where this way of thinking about it shows that the first is really the single condition.