2006 Part1 Q3a Comp

1. $\frac{\partial \vec{B}}{\partial t} = -\nabla\times\left[\alpha(\nabla\times\vec{B})\times\vec{B}\right]$ $\qquad \vec{B} = b_x(z)\hat{x} + b_y(z)\hat{y} + B_0\hat{z}$

linearized form: $\Rightarrow (\nabla\times\vec{B})\times\vec{B}_0$

$$\nabla\times\vec{B} = \hat{z}\frac{\partial}{\partial z}\times\left(b_x\hat{x}+b_y\hat{y}\right) = \frac{\partial b_x}{\partial z}\hat{y} - \frac{\partial b_y}{\partial z}\hat{x}$$

$$(\nabla\times\vec{B})\times\vec{B}_0 = B_0\left(\frac{\partial b_x}{\partial z}\hat{y} - \frac{\partial b_y}{\partial z}\hat{x}\right)\times\hat{z} = B_0\left(\frac{\partial b_x}{\partial z}\hat{x} + \frac{\partial b_y}{\partial z}\hat{y}\right)$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = -B_0\alpha\ \hat{z}\frac{\partial}{\partial z}\times\left(\frac{\partial b_x}{\partial z}\hat{x} + \frac{\partial b_y}{\partial z}\hat{y}\right) = -B_0\alpha\left(\frac{\partial^2 b_x}{\partial z^2}\hat{y} - \frac{\partial^2 b_y}{\partial z^2}\hat{x}\right)$$

$$\frac{\partial}{\partial t}\begin{pmatrix} b_x \\ b_y\end{pmatrix} = \alpha B_0\frac{\partial^2}{\partial z^2}\begin{pmatrix} b_y \\ -b_x\end{pmatrix} = \alpha B_0\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\frac{\partial^2}{\partial z^2}\begin{pmatrix} b_x \\ b_y\end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$$ A has eigenvalues $\pm i$

2. FTCS: $\vec{b} = \binom{b_x}{b_y}$ $\qquad \frac{\partial\vec{b}}{\partial t} = \alpha B_0 A\cdot\frac{\partial^2\vec{b}}{\partial z^2}$ $\qquad \vec{b}(z,t) = \vec{b}(z_j, t^n) = \vec{b}(j\delta z, n\delta t) \equiv \vec{b}_j^n$

$$\vec{b}_j^{n+1} = \vec{b}_j^n + \frac{\delta t\,\alpha B_0}{\delta z^2}A\cdot\left(\vec{b}_{j+1}^n - 2\vec{b}_j^n + \vec{b}_{j-1}^n\right)$$

Von-Neumann Stability Analysis: $\vec{b}_j^n \to r^n\tilde{b}_k e^{ij\theta}$ $\qquad \theta = k\delta z$

Let $s \equiv \frac{\delta t}{\delta z^2}\alpha B_0$

$$r\tilde{b}_k = \tilde{b}_k + sA\cdot\left(e^{i\theta}\tilde{b}_k - 2\tilde{b}_k + e^{-i\theta}\tilde{b}_k\right) = 2s(\cos\theta - 1)A\cdot\tilde{b}_k$$

Left-multiply by T where $TAT^{-1} = D$, a diagonal matrix. Define $T\tilde{b}_k \equiv \tilde{V}_k$

$\Rightarrow$ $r\tilde{V}_k = \tilde{V}_k + 2s(\cos\theta - 1)D\tilde{V}_k$.

Since D is diagonal, the equations uncouple. The two values for D are λ_A, the eigen values of A

$r = 1 + 2s(\cos\theta - 1)\lambda_A$.

$\lambda_A = \pm i$

Thus $r = 1 \pm i2s(\cos\theta - 1)$

$\Rightarrow |r|^2 > 1$, $\quad |r| > 1 \quad \Rightarrow$ always unstable

3. A BTCS method should have improved stability properties.

$$\frac{\vec{b}_j^{n} - \vec{b}_j^{n-1}}{\delta t} = \frac{\alpha B_0}{\delta z^2} A\cdot\left(\vec{b}_{j+1}^{n} - 2\vec{b}_j^{n} + \vec{b}_{j-1}^{n}\right)$$

or, letting $n \to n+1$,

$$\vec{b}_j^{n+1} = \vec{b}_j^{n} + \frac{\delta t}{\delta z^2}\alpha B_0 \; A\cdot\left(\vec{b}_{j+1}^{n+1} - 2\vec{b}_j^{n+1} + \vec{b}_{j-1}^{n+1}\right)$$

$$\vec{b}_j^{n+1} = \vec{b}_j^{n} + s\,A\cdot\left(\vec{b}_{j+1}^{n+1} - 2\vec{b}_j^{n+1} + \vec{b}_{j-1}^{n+1}\right)$$

stability analysis: $\vec{b}_j^{n} \to r^n \tilde{b}_k e^{ij\theta}$

$$r\tilde{b}_k = \tilde{b}_k + rs2(\cos\theta - 1)\,A\cdot\tilde{b}_k$$

left-multiply by $T_{...}$

$$r\tilde{V}_k = \tilde{V}_k + rs2(\cos\theta - 1)\,D\cdot\tilde{V}_k \qquad \text{eqns decouple...}$$

$$r = 1 - rs2(1-\cos\theta)\lambda_A$$

$$r\left(1 + 2s(1-\cos\theta)\lambda_A\right) = 1$$

$$r = \frac{1}{1 + 2s(1-\cos\theta)\lambda_A} \qquad \lambda_A = \pm i$$

$$r = \frac{1}{1 \pm i2s(1-\cos\theta)} \Rightarrow |r|^2 = \frac{1}{1 + 4s^2(1-\cos\theta)^2}$$

$|r|^2 < 1 \Rightarrow |r| < 1$ always stable for any timestep!