2007 I-2 GPP

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$$f(r,\bar{p}) = \frac{\hat{n}}{(2\pi m T)^{3/2}} \exp\left\{-\frac{(H - \Omega P_\theta)}{T}\right\}$$

a) $$\langle p_r \rangle = \int_{-\infty}^{\infty} p_r f(r,\bar{p})\, dp_r dp_\theta dp_z$$

$$\langle p_\theta \rangle = \int_{-\infty}^{\infty} p_\theta f(r,\bar{p})\, dp_r dp_\theta dp_z$$

$$\langle p_z \rangle = \int_{-\infty}^{\infty} p_z f(r,\bar{p})\, dp_r dp_\theta dp_z$$

Ω is the frequency of plasma rotation.

b) The factor $H - \Omega P_\theta$ can be rewritten as follows:

$$H - \Omega P_\theta = \frac{1}{2m}(p_r^2 + p_\theta^2 + p_z^2) - \Omega r p_\theta + \frac{\Omega w_c m r^2}{2} =$$

completing the square in p_θ

$$= \frac{1}{2m}\left[p_r^2 + p_z^2 + (p_\theta - mr\Omega)^2\right] + \frac{mr^2}{2}\left[\Omega w_c - \Omega^2\right] - e\varphi(r) =$$

$$= \frac{1}{2m}\left[p_r^2 + p_z^2 + (p_\theta - mr\Omega)^2\right] + \psi(r)$$

Thus, $n(r) = \int f(\bar{p}, r)\, d^3p =$

$$= \frac{\hat{n}}{(2\pi m T)^{3/2}} \exp\left\{-\frac{\psi(r)}{T}\right\} \int_{-\infty}^{\infty} dp_r \exp\left(\frac{-p_r^2}{2mT}\right) \int_{-\infty}^{\infty} dp_\theta \exp\left(-\frac{(p_\theta - mr\Omega)^2}{2mT}\right) \cdot$$

$$\cdot \int_{-\infty}^{\infty} dp_z \exp\left(\frac{-p_z^2}{2mT}\right) =$$

$$= \hat{n}\exp\left\{\frac{-\Psi(r)}{T}\right\} = n(r)$$; integrals over p_r, p_z, p_θ cancel the normalization factor $\frac{1}{(2\pi m T)^{3/2}}$

c) Assume that for small r, $n(r) \simeq \hat{n}$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) \simeq 4\pi e\hat{n}$$

$$r\frac{\partial\varphi}{\partial r} = \frac{4\pi e\hat{n}r^2}{2}$$

$$\varphi = \frac{4\pi e\hat{n}r^2}{4} = \boxed{\frac{1}{4}\frac{m}{e}w_p^2 r^2}$$

d) $n(r) = \hat{n}\exp\left\{\frac{-\Psi(r)}{T}\right\}$; For $n(r)$ to be monotonically decreasing, $\Psi(r)$ has to be monotonically increasing;

$$\frac{\partial\Psi}{\partial r} > 0$$

Taking $\Phi(r) \simeq \frac{1}{4}\frac{m}{e}w_p^2 r^2$,

$$\frac{\partial\Psi}{\partial r} = mr\left(\Omega w_c - \Omega^2\right) - \frac{1}{2}m w_p^2 r > 0$$

$$\boxed{\left(\Omega w_c - \Omega^2\right) - \frac{1}{2}w_p^2 > 0}$$

e) $\Omega^2 - \Omega w_c + \frac{1}{2} w_p^2 < 0$

$\left(\Omega - \frac{w_c}{2}\right)^2 - \frac{w_c^2}{4} + \frac{1}{2} w_p^2 < 0$

$\left(\Omega - \frac{w_c}{2}\right)^2 < \frac{w_c^2}{4} - \frac{1}{2} w_p^2$

- Max Ω: $\left(\Omega - \frac{w_c}{2}\right)^2 = \frac{w_c^2}{4} - \frac{1}{2} w_p^2$

$$\boxed{\Omega_{max} = \sqrt{\frac{w_c^2}{4} - \frac{1}{2} w_p^2} + \frac{w_c}{2}}$$

- Min Ω: $\left(\Omega + \frac{w_c}{2}\right)^2 = \frac{w_c^2}{4} - \frac{1}{2} w_p^2$

(assume $\Omega < 0$)

$$\boxed{\Omega_{min} = \sqrt{\frac{w_c^2}{4} - \frac{1}{2} w_p^2} - \frac{w_c}{2}}$$

$\left(\frac{w_p^2}{w_c^2}\right)_{max}$ $\quad \frac{w_c^2}{4} - \frac{1}{2} w_p^2 > 0 \longrightarrow \boxed{\frac{w_p^2}{w_c^2} < \frac{1}{2}}$

- when $\frac{w_p^2}{w_c^2} = \frac{1}{2}$, $\boxed{\Omega = \frac{w_c}{2}}$