

1999 Part 1 Q4A

Comp

$$1. V_j^{n+1} = V_j^n + \frac{\delta t}{2\delta x} [P_{j+1}^n - P_{j-1}^n]$$

$$P_j^{n+1} = P_j^n + \frac{c^2 \delta t}{2\delta x} [V_{j+1}^n - V_{j-1}^n]$$

Stability Analysis: let  $V_j^n = r^n V_k e^{ij\theta_k}$ ,  $P_j^n = r^n P_k e^{ij\theta_k}$ , let  $s \equiv \frac{\delta t}{\delta x}$

$$\Rightarrow r V_k = V_k + i s \sin \theta_k P_k$$

$$r P_k = P_k + i s c^2 \sin \theta_k V_k \rightarrow P_k = \frac{i s c^2 \sin \theta_k}{r-1} V_k$$

$$r V_k = V_k - \frac{s^2 c^2 \sin^2 \theta_k}{r-1} V_k$$

$$\Rightarrow r(r-1) = r-1 - s^2 c^2 \sin^2 \theta_k$$

$$r^2 - r = r-1 - s^2 c^2 \sin^2 \theta_k$$

$$r^2 - 2r + 1 + s^2 c^2 \sin^2 \theta_k = 0$$

$$\Rightarrow r = \frac{2 \pm \sqrt{4 - 4 - 4s^2 c^2 \sin^2 \theta_k}}{2} = 1 \pm \sqrt{-s^2 c^2 \sin^2 \theta_k}$$

$$r = 1 \pm i s c \sin \theta_k$$

$$\Rightarrow |r| > 1 \text{ always} \Rightarrow \text{unstable}$$

2.  $\theta$ -implicit method:

$$V_j^{n+1} = V_j^n + \frac{(1-\theta)}{2} s [P_{j+1}^n - P_{j-1}^n] + \frac{\theta}{2} s [P_{j+1}^{n+1} - P_{j-1}^{n+1}]$$

$$P_j^{n+1} = P_j^n + \frac{(1-\theta)}{2} c^2 s [V_{j+1}^n - V_{j-1}^n] + \frac{\theta}{2} c^2 s [V_{j+1}^{n+1} - V_{j-1}^{n+1}]$$

$$r V_k = V_k + i(1-\theta) s \sin \theta_k P_k + i r \theta s \sin \theta_k P_k$$

$$r P_k = P_k + i(1-\theta) c^2 s \sin \theta_k V_k + i r \theta c^2 s \sin \theta_k V_k$$

$$\hookrightarrow P_k = \frac{i c^2 s \sin \theta_k (1-\theta + r\theta)}{r-1} V_k$$

$$r-1 = \frac{i s \sin \theta_k (1-\theta + r\theta) \cdot i c^2 s \sin \theta_k (1-\theta + r\theta)}{r-1}$$

$$(r-1)^2 = -s^2 c^2 \sin^2 \theta_k (1-\theta + r\theta)^2$$

Take  $\theta=1$  for simplicity:

$$(r-1)^2 = -s^2 c^2 \sin^2 \theta_k r^2$$

$$r^2 - 2r + 1 = -s^2 c^2 \sin^2 \theta_k r^2$$

$$r^2(1 + s^2 c^2 \sin^2 \theta_k) - 2r + 1 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(1 + s^2 c^2 \sin^2 \theta_k)}}{2(1 + s^2 c^2 \sin^2 \theta_k)}$$

$$r = \frac{1 \pm i s c \sin \theta_k}{1 + s^2 c^2 \sin^2 \theta_k}$$

$$|r|^2 = \frac{1 + s^2 c^2 \sin^2 \theta_k}{1 + 2s^2 c^2 \sin^2 \theta_k + (s^2 c^2 \sin^2 \theta_k)^2}$$

$\Rightarrow |r|^2 < 1$  for all  $s$   
unconditionally stable