Abstract

This file contains solutions to the problem sets from the past preliminary exams on physics. The formulations of the problem have not been included, but can be found on the official physics department site. The file has not even been edited yet, so its very raw state is just for the sake of an attempt to make it available publicly as soon as possible.

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In this new version the following solutions are updated: m00m3,m02j1,e02j2,q02m1,s99j1,s01j2,s02j2,s02j3,s03m1,s03m2,s03m3.
We thank everybody who sent us the comments.
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Contents

1 Classical Mechanics

1.1 m/m98j1 V

The action with Lagrange term for constraint included is

$$\int_{-w}^{w} y\sqrt{1+\dot{y}^2}dx - \lambda\left(\int_{-w}^{w}\sqrt{1+\dot{y}^2}dx - L\right) \tag{1}$$

The conserved quantity, charge corresponding to the translations along x axis is

$$H = p\dot{y} - L, \qquad p = \frac{\delta L}{\delta \dot{y}}, \tag{2}$$

so

$$p = (y-\lambda)\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}. \tag{3}$$

Up to the constant term,

$$H = -(y-\lambda)\frac{1}{\sqrt{1+\dot{y}^2}}. \tag{4}$$

After expressing $\dot{y} = f(y)$ and integrating, one gets

$$y = \lambda + H\cosh\frac{x}{H}, \tag{5}$$

and the length condition

$$\int_0^w \sqrt{1+\dot{y}^2} = \int_0^w \cosh\frac{x}{H}dx = H\sinh\frac{w}{H} = L \tag{6}$$

1.2 m/m98j2 V

Let the first particle is at distance r and the angle α from the center of mass (its motion decouples). Then, the Lagrangian is

$$L = 2\frac{m}{2}[r^2\dot{\alpha}^2 + \dot{r}^2] - \frac{k}{2}(2r-2r_0)^2 \tag{7}$$

The momentums are

$$p_r = 2m\dot{r}, \qquad p_{al} = 2mr^2\dot{\alpha} \tag{8}$$

and the Hamiltonian is

$$H = \frac{p_r^2}{4m} + \frac{p_\alpha^2}{4mr^2} + \frac{1}{2}4k(r - r_0)^2 \tag{9}$$

The angular momentum p_α conserves and equations for α decouple

$$p_\alpha = \text{const}, \qquad \dot{\alpha} = \frac{\partial H}{\partial p_\alpha} = \frac{1}{2mr^2}p_\alpha \tag{10}$$

while for

a. radial component

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{1}{2m}p_r \tag{11}$$

$$2m\ddot{r} = \dot{p}_r = -\frac{\partial H}{\partial r} = 2\frac{p_\alpha^2}{4mr^3} - 4k(r - r_0) \tag{12}$$

b. The system will oscillate between two turning points. They first one is just the initial point, the second is determined from equation for vanishing kinetic energy in that point

$$\frac{p_\alpha^2}{4mr_0^2} = \frac{p_\alpha^2}{4mr^2} + 2k(r - r_0)^2 \tag{13}$$

Form small v one has small p_α and as a result

$$\Delta l = 2(r - r_0) = \frac{1}{2}\frac{mv^2}{kr_0}, \tag{14}$$

where $r_0 = l/2$, and v is initial velocity in the lab frame.

1.3 m/m98j3 V

The equations of motion are

$$\ddot{x} = g - N\sin\omega t \tag{15}$$

$$\ddot{y} = N\cos\omega t \tag{16}$$

$$\dot{y} = \dot{x}\tan\omega t \tag{17}$$

With an anzatz $N = A\sin\omega t$ one gets

$$\ddot{x} = g - A\sin^2\omega t \tag{18}$$

$$\ddot{y} = A\cos\omega t\sin\omega t \tag{19}$$

$$\dot{y} = \dot{x}\tan\omega t \tag{20}$$

Then,

$$\dot{x} = gt - \frac{1}{2}At + \frac{A}{4\omega}\sin 2\omega t \tag{21}$$

and

$$\dot{y} = \frac{A}{4\omega}(1 - \cos 2\omega t) \tag{22}$$

All equations are satisfied with $A = 2g$ and

$$x = \frac{g}{4\omega^2}(1 - \cos 2\omega t) \tag{23}$$

$$y = \frac{g}{4\omega^2}(2\omega t - \sin 2\omega t) \tag{24}$$

That is cycloid.

1.4 m/m98m1 V

From equation for forces at an infinitesimal patch of circle $dT = -\mu T d\phi$ follows $T = T_0 e^{-\mu\phi} = mge^{-\mu\frac{\pi}{2}}$.

1.5 m/m98m2 V

We consider case, when the horizontal line lies in the same vertical plane as a circle, below it.

$$L = \frac{1}{2}m\left((r\dot{\alpha})^2 + \dot{x}^2\right) + mgr\cos\alpha - \frac{1}{2}k\left((x - r\sin\alpha)^2 + (a - r\cos\alpha)^2\right) \tag{25}$$

for small deflections, where $y = r\alpha$, gives the canonical kinetic term and the following bilinear form for the potential term

$$\begin{pmatrix} k & -k \\ -k & \frac{ka+mg}{r} \end{pmatrix} \tag{26}$$

with the eigenvalues

$$\lambda = \frac{1}{2}\left(-(k+k_1) \pm \sqrt{(k+k_1)^2 - 4(kk_1 - k^2)}\right), \tag{27}$$

where $k_1 = \frac{ka+mg}{r}$ and frequencies $\omega^2 = \frac{\lambda}{m}$.

1.6 m/m98m3 V

a.

$$v = \sqrt{2gh} = 4.5m/s^2 \tag{28}$$

b. Balloon is inclined in the direction of the acceleration with the angle $\tan\theta = \frac{a}{g} = 0.1$. c.

$$\frac{Mg}{S} = \rho g H, \qquad H = \frac{M}{\rho S} = 3\,cm \tag{29}$$

d.

$$\Delta P = 4\frac{\sigma}{R} = 20Pa \tag{30}$$

e. Since

$$\eta \nabla v = \frac{1}{\rho}\nabla p \tag{31}$$

and $\sum v_i = 0$, where v_i are incoming velocities,

$$\sum(p_i - p_{center}) = 0. \tag{32}$$

Then

$$p_{center} = \frac{1}{4}(2\alpha + 1)p \tag{33}$$

it should be greater than p for the flow to go out. Therefore

$$\alpha > \frac{3}{2} \tag{34}$$

1.7 m/m99j1 T

m99j2 $\frac{F}{Mg} = tg\theta$

$R = b - a\sin\theta$

$Mw^2R = F$ and $T = \frac{2\pi}{w}$ Thus $T = 2\pi\sqrt{\frac{b - a\sin\theta}{gtg\theta}}$

1.8 m/m99j2 T

m99j2 $r^2 = x^2 + y^2$ Lagrangian is

$$L = \frac{m}{2}[r^2\dot{\phi}^2 + \dot{r}^2] - mg\sqrt{r^2 - R^2} \tag{35}$$

Equation of motion $mr^2 w = L = const$ ($\dot{\phi} = w$) and

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{mgr}{\sqrt{r^2 - R^2}} \tag{36}$$

The stationary orbit is when $\dot{r} = 0$ and $r^2 = r_0^2 = R^2 + g^2 w^{-4}$ Equation of motion for small fluctuation $\xi = r - r_0$ is $\ddot{\xi} + \Omega^2 \xi = 0$

$$\Omega^2 = w^2(3 - \frac{R^2 w^4}{g^2}) \tag{37}$$

The oscillations are unstable if $\Omega^2 < 0$ or when (here we substitute expression for w from formula for r_0)

$$r_0^2 < \frac{4}{3}R^2 \tag{38}$$

1.9 m/m99j3 T

m99j2 Forces acting on the element of rod from x to $x + dx$ ($x \in [-L/2..L/2]$) all together gives zero result:

$$T(x + dx) + \frac{dxm(R + x)w^2}{L} = T(x) + \frac{dxmMG}{L(R + x)^2} \tag{39}$$

After integrating one gets

$$T(x) = -\frac{mMG}{L(R + x)} - \frac{mx(R + x/2)w^2}{L} + C \tag{40}$$

The condition that total gravitational force is equal to total centrifugal force is $T(L/2) = T(-L/2)$. Solving this constraint we will find $w^2 = MG/R(R^2 - L^2/4)$, and after substituting this back we will have thee following result for $T(x)$

$$T(x) = -\frac{mMG}{L(R + x)} - \frac{mMGx(R + x/2)w^2}{LR(R^2 - L^2/4)} + C \tag{41}$$

Now we will use condition that at the end of the rod there should not be any tension (end of the rod is massless). Or $T(L/2) = T(-L/2) = 0$. This gives

$$C = \frac{mMG(R^2 + L^2/8)}{RL(R^2 - L^2/4)} \tag{42}$$

Now we can calculate $T(0)$

$$T(x = 0) = \frac{3mMGL}{8R(R^2 - L^2/4)} \tag{43}$$

1.10 m/m99m1 V

From conservation of an angular momentum there is a rotating frame in which ball moves only in the vertical direction.

Let the frame 'bar' rotate with Ω counter clockwise.

$$\omega_x = \omega_{\bar{x}} \cos \Omega t - \omega_{\bar{y}} \sin \Omega t \tag{44}$$

$$\omega_y = \omega_{\bar{x}} \sin \Omega t + \omega_{\bar{y}} \cos \Omega t \tag{45}$$

(the same relation between any 'bar' and lab vector components). Then acceleration-force equations of motion in lab frame:

$$m\dot{v}_{\bar{y}} = 0 \tag{46}$$
$$m(\dot{v}_{\bar{x}} - \Omega^2 r) = N_{\bar{x}} \tag{47}$$
$$m\dot{v}_z = -mg + N_z \tag{48}$$

and angular momentum-angular acceleration equations of motion in lab frame

$$J\dot{\omega}_z = rN_{\bar{y}} \tag{49}$$
$$J\dot{\omega}_x = r\sin\Omega t\, N_z \tag{50}$$
$$J\dot{\omega}_y = -r\cos\Omega t\, N_z \tag{51}$$

can be rewritten into 'bar' frame

$$J\dot{\omega}_z = rN_{\bar{y}} \tag{52}$$
$$J(\dot{\omega}_{\bar{x}} - \Omega\omega_{\bar{y}}) = 0 \tag{53}$$
$$J(\dot{\omega}_{\bar{y}} + \Omega\omega_{\bar{x}}) = -rN_z \tag{54}$$

Since $N_{\bar{y}} = 0$

$$\dot{\omega}_z = 0, \qquad \omega_z = -\Omega\frac{R}{r}. \tag{55}$$

Substituting $v_z = r\omega_{\bar{y}}$ in the (??) one gets

$$J(\dot{\omega}_{\bar{x}} - \Omega\omega_{\bar{y}}) = 0 \tag{56}$$
$$J(\dot{\omega}_{\bar{y}} + \Omega\omega_{\bar{x}}) = -r(mr^2\dot{\omega}_{\bar{y}} - mgr) \tag{57}$$

and then

$$\bar{J}\ddot{\omega}_{\bar{y}} + J\Omega^2\omega_{\bar{y}} = 0, \qquad \bar{J} = J + mr^2 \tag{58}$$

with oscillating solution

$$\omega_{\bar{y}} = B\sin\xi t, \tag{59}$$

where $\xi^2 = \Omega^2\frac{J}{J+mr^2}$. From $\omega_{\bar{x}}(0) = 0$ one gets

$$B = -\frac{mgr}{(J+mr^2)\xi} \tag{60}$$

which determines

$$v_z = r\omega_{\bar{y}}\sin\xi t = -\frac{mgr^2}{(J+mr^2)\xi}\sin\xi t \tag{61}$$

and

$$z = -\frac{mgr^2}{(J+mr^2)\xi^2}(1-\cos\xi t) = -\frac{5}{2}\frac{g}{\Omega^2}(1-\cos\xi t) \tag{62}$$

The ball will be oscillating between two horizontal lines.

1.11 m/m99m2 V

The potential $\frac{1}{r}^2$ has the same form as an effective potential for a particle moving along radius due to its angular momentum. Therefore (for the effective scattering particle, in the central coordinate system, where $\phi = \pi$ is zero scattering), given

$$\frac{m}{2}(\dot{r}^2 + \frac{M_0^2}{m^2 r^2} + \frac{\beta}{r^2}) = E_0 \tag{63}$$

and rewriting

$$\phi = \int \frac{M_0 dr}{\sqrt{\frac{2E_0}{m} - \frac{M_0^2 + \beta m^2}{r^2}}} \frac{1}{mr^2} \tag{64}$$

one can immediately find the answer without integrating, just from comparison with a freely moving particle

$$\phi = \pi \sqrt{\frac{M_0^2}{M_0^2 + m\beta^2}} \tag{65}$$

(Here M_0 and m are for the effective particle, $m = m_{eff} = \frac{m_1 m_2}{m_1 + m_2}$ and $M_0 = bvm_{eff}$, and $m_{eff} = \frac{2}{3}M$).

The incoming light particle (which is 2 times lighter), will scatter in the lab frame at the angle

$$\tan\theta = \frac{\sin\phi}{-\frac{1}{3} + \frac{2}{3}\cos(\pi - \phi)} \tag{66}$$

1.12 m/m99m3 V

a. From the functional for the energy

$$L = \int 2\pi x dx (\tau\sqrt{1 + (y')^2} + \rho g y) \tag{67}$$

in the limit $y' \to 0$ one gets the stationary equation

$$\frac{d}{dx}(x\tau y') = x\rho g, \tag{68}$$

with solution

$$y(x) = \frac{x^2 \rho g}{4\tau} \tag{69}$$

b. Including dependence on time in lagrangian with zero angular mode, one gets

$$L = \int 2\pi x dx \frac{1}{2}(\rho(\partial_t y)^2 - \tau(\partial_x y)^2) \tag{70}$$

with the standard equations of motion for cylindrically symmetrical waves

$$\partial_t(x\rho\partial_t y) - \partial_x(x\tau y) = 0 \tag{71}$$

Plugging $y(x,t) = y(x)e^{i\omega t}$ one gets

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \Omega^2 y = 0 \tag{72}$$

where $\Omega^2 = \frac{\rho\omega^2}{\tau}$

Its solutions are given by Bessel functions. With $x = \frac{1}{\Omega}\tilde{x}$ the equation becomes canonical and the answer is $y = J_0(\Omega x)$. Let c_0 is the first root $J_0(c_0) = 0$, then

$$\omega = \sqrt{\frac{\tau c_0^2}{\rho R^2}} \tag{73}$$

1.13 m/m00j1 T

m00j1

Let $\alpha(x)$ be the angle between the tangential line to ark (at point x) and horizontal line. Obviously $tg\alpha = y'$. Then the mass of the element of ark from x to $x+dx$ is $dm = \mu dl = \frac{\mu dx}{\cos\alpha}$. Let us $\overrightarrow{N}$ be the normal (no gluing-no tangential) force in arc. Then $\overrightarrow{N} = \hat{x}N\cos\alpha + \hat{y}N\sin\alpha$. The element of arc is in rest. As a consequences the sum of all forces acting on it is zero. These forces are: normal force at the one end (at x), normal force at another end (at $x+dx$) and gravitational force.

$$N\cos\alpha(x+dx) - N\cos\alpha(x) = 0 \tag{74}$$

$$N\sin\alpha(x+dx) - N\cos\alpha(x) + dmg = 0 \tag{75}$$

Or

$$N\sin\alpha(x) = C = const \tag{76}$$

and thus

$$(Ctg\alpha)' + \frac{\mu g}{\cos\alpha} = 0 \tag{77}$$

As a result

$$-\frac{\mu g dx}{C} = \frac{d\alpha}{\cos\alpha} \tag{78}$$

and then

$$-\frac{\mu g dx}{C} = \frac{d(y')}{\sqrt{1+(y')^2}} \tag{79}$$

Integrating this we get

$$y' = sh(\frac{\mu g}{C}(x-x_0)) \tag{80}$$

At last

$$y' = -\frac{C}{\mu g}csh(\frac{\mu g}{C}(x-x_0)) \tag{81}$$

Arc has form of so-called chain-line (csh). C has sence of horizontal force in arc.

1.14 m/m00j2 T

m00j2 Let $\varphi(t) = \pi - \theta(t)$ and $\varphi_0 = \varphi(t=0)$ And also let $l \in [0..L]$ be the coordinate along one rod, such as $l=0$ is a free end (on the floor) and $l=L$ is a tied end. Then the coordinates are

$$x(l) = (L-l)\cos\varphi \tag{82}$$

$$y(l) = l\sin\varphi \tag{83}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{\varphi}^2[l^2 + \sin^2\varphi(L^2 - 2Ll)] \tag{84}$$

The kinetic energy of one rod is

$$E = \int_0^L \frac{dlm}{2L}v^2 = \frac{mL^2\dot{\varphi}^2}{6} \tag{85}$$

Energy conservation low says

$$\frac{mL^2\dot{\varphi}^2}{6} + \frac{mgL\sin\varphi}{2} = \frac{mgL\cos\theta}{2} \tag{86}$$

a). The moment just before the rods touch the floor is when $\varphi = 0$. This time $\dot{x} = 0$ since it is proportional to $\sin\varphi$. Thus the velocity is vertical and equal

$$L\dot{\varphi}\cos\varphi = \sqrt{3gL\cos\theta} \tag{87}$$

b). Now we will find the force between the tied ends. Due to the symmetry these force (F) is horizontal, and this is only one horizontal force acting on the rod. According to the seconds Newton's low, applied to the center of mass of the rod

$$F(t) = m\ddot{x}(t) \tag{88}$$

We know the dependence $x(t) = x(\varphi(t))$ and $\dot{\varphi}(t) = \dot{\varphi}(\varphi(t))$ (energy conservation low). Thus $\ddot{(t)} = \dot{\varphi}^2 \frac{d^2}{d\varphi^2}x(\varphi) + \frac{d}{d\varphi}x(\varphi)\frac{d}{d\varphi}\dot{\varphi}(\varphi)\dot{\varphi}$ The second term gives no contribution since $\frac{d}{d\varphi}x(\varphi)$ proportional to $\sin\varphi$ and vanishes when $\varphi = 0$. And the contribution of the first one is

$$\ddot{x}(\varphi = 0) = \frac{3g}{2}\cos\theta \tag{89}$$

Or, finally

$$F = \frac{3mg}{2}\cos\theta \tag{90}$$

1.15 m/m00j3 T

m00j3

Let r be the distance between the hole in the table and mass m_2. Then the largarngian is

$$L = \frac{1}{2}[(m_1 + m_2)\dot{r}^2 + m_2 r^2\dot{\varphi}^2] - m_2 g r \tag{91}$$

Equations of motion are

$$m_1 r^2\dot{\varphi} = L_\varphi = const \tag{92}$$

$$(m_1 + m_2)\ddot{r} = -m_2 g + \frac{L_\varphi^2}{m_1 r^3} \tag{93}$$

The equilibrium position is $r_0 = \frac{m_2 g}{m_1 w^2}$, where $w = \dot{\varphi}$. We want to consider small oscillations of variable $\xi = r - r_0$. Expanding up to the first order in ξ the equation of motion for $r$we will get

$$\ddot{\xi} + \Omega^2\xi = 0 \tag{94}$$

where

$$\Omega^2 = \frac{3L_\varphi^2}{m_1(m_1 + m_2)r_0^4} = \frac{3m_1}{m_1 + m_2}w^2 \tag{95}$$

We know that the orbit is closed ($\Rightarrow \Omega = wn$)and there is only one minimum (maximum) per period ($\Rightarrow n = 1$). Thus $m_2 = 2m_1$

1.16 m/m00m1 V

The potential is (a)

$$GM\frac{1}{2\pi}\int\frac{d\theta}{|R + re^{i\theta}|} = GM\frac{1}{2\pi}\frac{1}{R}\int d\theta(1 + 2x\cos\theta + x^2)^{-1/2} = \frac{GM}{R}\left(1 + \frac{1}{4}x^2\right) + O(x^3), \tag{96}$$

where $x = \frac{r}{R}$, and (b)

$$\omega_0^2 R = \frac{GM}{R^2}\left(1 + \frac{3}{4}\frac{r^2}{R^2}\right) \tag{97}$$

To find small oscillations consider the Hamiltonian form with

$$p_\phi = mR^2\dot{\phi}, \qquad p_r = m\dot{R}, \tag{98}$$

and

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mR^2} - \frac{GMm}{R}\left(1 + \frac{1}{4}\frac{r^2}{R^2}\right) \tag{99}$$

The stationary point for r motion corresponds to

$$\frac{p_\phi^2}{mR_0^3} = \frac{GMm}{R_0^2}\left(1 + \frac{3}{4}\frac{r^2}{R_0^2}\right) \tag{100}$$

Plugging it in the effective potential term, and taking the second derivative, one finds

$$\frac{\partial^2 V}{\partial R^2} = \frac{GMm}{R^3}\left(1 - \frac{3}{4}\frac{r^2}{R^2}\right) \tag{101}$$

Then,

$$\frac{\omega_r}{\omega_\phi} = 1 - \frac{3}{4}x^2 \tag{102}$$

and

$$\Delta\phi = 2\pi\frac{3}{4}x^2 \tag{103}$$

1.17 m/m00m2 V

Let θ is an angle between the line from the corner to the center of the ladder and the floor. Then from conservation of energy

$$\frac{m}{2}\dot{\theta}^2 r^2 + \frac{J}{2}\dot{\theta}^2 = mgr(\sin\theta_0 - \sin\theta) \tag{104}$$

one find the relation between $\dot{\theta}$ and θ, and also equation of motion for θ

$$\ddot{\theta} = -\frac{mgr}{J_{eff}}\cos\theta \tag{105}$$

The x coordinate of the center of mass moves with an acceleration

$$\ddot{x} = -\ddot{\theta} r\sin\theta - \dot{\theta}^2 r\cos\theta \tag{106}$$

When the ladder separates from the wall $\ddot{x} = 0$, thus

$$\frac{mgr}{J_{eff}} r\cos\theta_c\sin\theta_c - \dot{\theta}_c^2 r\cos\theta_c = 0 \tag{107}$$

and the result for critical angle is

$$\sin\theta_c = \frac{2}{3}\sin\theta_0 \tag{108}$$

1.18 m/m00m3 T

Lgrangian of our system is

$$L = \frac{m}{2}[R^2\dot{\theta}^2 + w^2R^2\sin^2\theta] + mgR\cos\theta \tag{109}$$

Using that $w^2 = \frac{g}{R}$ we obtain the system with effective potential energy of the form

$$U = -\frac{g}{R}[\cos\theta + \frac{1}{2}\sin^2\theta] \tag{110}$$

and with kinetic energy $\frac{1}{2}\dot{\theta}^2$. Now the period of oscillations with amplitude θ is

$$T(\theta) = 4\sqrt{\frac{R}{2g}}\int_0^\theta \frac{dt}{\sqrt{\cos t - \cos\theta + \frac{1}{2}(\sin^2 t - \sin^2\theta)}} \tag{111}$$

Expanding near zero, treating $t, \theta << 1$ we get

$$T(\theta) = \frac{8}{\theta}\sqrt{\frac{R}{g}}\int_0^1 \frac{dt}{\sqrt{1-t^4}} \tag{112}$$

1.19 m/m01j1 T

m01j3 As usual we use cylindrical coordinates and

$$L = \frac{m}{2}[\dot{r}^2 + r^2\dot{\phi}^2] - mgz \tag{113}$$

Here $z = -b\cos\psi$ and $r = a + b\sin\psi$ Angular moment conservation low gives

$$mr^2w = L + const \tag{114}$$

where $w = \dot{\phi}$. Circular orbit corresponds to the angle ψ_0 such as $tg\psi_0 = \frac{w^2r(\psi_0)}{g}$. Equation of motion in the terms of r is

$$m\ddot{r} = \frac{L^2}{mr^3} - m\frac{dz}{dr} \tag{115}$$

And frequency of small oscillations is

$$\Omega^2 = 3w^2 - \frac{d^2z}{dr^2} = 3w^2 - \frac{g}{b\cos^3\psi_0} \tag{116}$$

1.20 m/m01j2 T

m01j2 The most important quantity in this problem is the angle between line, which tangential to the Earth's surface and the line, which is tangential to the radius. If we will drop second order in ϵ we can treat $\frac{dr(\phi)}{rd\phi}$ also as a small quantity and drop it's second order as well. Then this angle α is just equal to $\alpha = -\frac{dr(\phi)}{rd\phi}$ (Minus here is due to the sign of $d\phi$). The centrifugal force on the surface is

$$F = mw^2r(\phi)\cos\phi \tag{117}$$

and directed to the axis of Earth's rotation. Together with gravitational force mg, directed along the radius this force should form the force just perpendicular to the surface. Using "sin" theorem we get

$$\frac{\sin\alpha(\phi)}{F} = \frac{\sin(\phi + \alpha(\phi))}{mg} \tag{118}$$

(note that here g depends on ϕ). We will drop second order of α and the natural dimensionless small parameter will be $\frac{w^2r_e}{g(r_e)}$. Expanding in this parameter we can neglect the dependence in g on ϕ and α in $\sin(\phi + \alpha(\phi))$. After integrating we will have

$$r = r_e(1 - \frac{w^2r_e\sin^2\phi}{2g(r_e)} + O((\frac{w^2r_e}{g(r_e)})^2)) \tag{119}$$

Thus $\epsilon = \frac{w^2r_e}{2g(r_3)} = \frac{3w^2}{8\pi G\rho}$

1.21 m/m01j3 T

m01j3

Let the (constant in time) angle between conserved angular momentum and axis of Earth be ψ, angular frequency of rotation is w and precession frequency is Ω. Since no forces act on the Earth angular momentum

$$I(t)_{ij} = (Omega(t) + w(t))^j = M_i = const \tag{120}$$

Let us consider moment t. Let choose the cordinate system in the way that M has only z projection and center of Earth has zero y coordinate. Then

$$M = \{0, 0, M\} \tag{121}$$

$$\Omega = \{0, 0, \Omega\} \tag{122}$$

$$w = \{w \sin\psi, 0, w\cos\psi\} \tag{123}$$

In the coordinate frame , corresponding to Earth

$$\Omega = \{-\Omega\sin\psi, 0, \Omega\cos\psi\} \tag{124}$$

$$w = \{0, 0, w\} \tag{125}$$

In this frame

$$I(\Omega + w) = \{-I_{xx}\Omega\sin\psi, 0, I_{zz}(\Omega\cos\psi + w)\} \tag{126}$$

Going back to the inertial frame ,

$$I(\Omega + w) = \{-I_{xx}\Omega\sin\psi\cos\psi + I_{zz}(\Omega\cos\psi + w)\sin\psi, 0, \cos\psi I_{zz}(\Omega\cos\psi + w) + I_{xx}\Omega\sin^2\psi\} \tag{127}$$

Now we have to satisfy our assumption about direction of M. As a consequences

$$0 = -I_{xx}\Omega\sin\psi\cos\psi + I_{zz}(\Omega\cos\psi + w)\sin\psi \tag{128}$$

And

$$\Omega = \frac{-w}{\epsilon\cos\psi} \tag{129}$$

where

$$\epsilon = \frac{I_{zz} - I_{xx}}{I_{zz}} \tag{130}$$

Remark (V)

1. If one expresses the answer in terms of L — angular momentum, the result is $\Omega = \frac{L}{I_x}$. What is called by the angular velocity of precession — is the angular velocity of rotation of the axis of the current angular velocity around the fixed direction in space along the angular momentum (which remains the same) in the inertial coordinate system (relative to the stars). The precession in this sense is slow when $I_x \gg I_z$ (rod) and is approximately the same as the current angular speed of rotation for a sphere-like object $I_x \sim I_z$.

2. By guess, in the problem they ask about estimation of some other quantity (what is confusing with common definition) — the angular velocity of rotation of the axis of the current angular velocity around the fixed direction attached to the Earth, that is — in rotating coordinate system of the Earth. From conservation of the angular momentum L written in the rotating x, y, z-coordinate system attached to the Earth we have

$$\dot{L} + \omega \times L = 0 \tag{131}$$

where ω is the current angular velocity (as physical object it is defined with respect to the coordinate system of stars, and by definition it is the angular velocity of rotating x, y, z-frame), but *expanded in the* x, y, z*-Earth connected* coordinate system. That results

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z, \qquad \dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_z\omega_x, \qquad \dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_x\omega_y, \tag{132}$$

For $O(2)$ symmetrical body we have $I_x = I_y$, and thus $\omega_z(t) = \text{const} \equiv \omega_{z0}$ and the remaining equations become linear

$$\ddot{\omega}_x + (\varepsilon\omega_{z0})^2\omega_x = 0 \tag{133}$$

with the harmonic solution having frequency $\tilde{\Omega} = \varepsilon\omega_{z0}$, which describes slow rotating of the current axis of angular speed around the z-axis direction (which is attached to the Earth!, and which is itself rotates with approximately 1-day period around the fixed direction of the angular momentum relative to the stars!!).

1.22 m/m01m1 T

a).

$$0 = dF = dx\rho g + T(x + dx) - T(x) \tag{134}$$

Thus

$$\frac{\partial T}{\partial x} = -\rho g \tag{135}$$

But

$$T(x) = KL(\frac{\partial S}{\partial x} - 1) \tag{136}$$

Result:

$$S'' = -\frac{\rho g}{KL} \tag{137}$$

Or

$$S(x) = a + bx + cx^2/2 \tag{138}$$

From the equation we have $c = -\frac{\rho g}{KL}$. Since $S(0) = 0$ $a = 0$. We also know that $T(L) = 0$ and thus

$$b - \frac{\rho g}{K} - 1 = 0 \tag{139}$$

Finally

$$S_0(x) = x + \frac{\rho g}{K}[x + \frac{x^2}{2L}] \tag{140}$$

b).The wave equation (as usual) has form

$$S'' - v^2\ddot{S} = 0 \tag{141}$$

where $v^2 = \frac{KL}{\rho}$. The general solution is

$$S(x, t) = f(x - vt) + g(x + vt) \tag{142}$$

At the moment $t = 0$ $S(x, 0) = S_0(x)$. Hence

$$f(x) + g(x) = S_0(x) \tag{143}$$

(here $x > 0$). Also at the moment $t = 0$ velocity is equal to zero:

$$f'(x) - g'(x) = 0 \tag{144}$$

for $x > 0$. Result: for $x > 0$

$$f(x) = g'(x) = \frac{1}{2} S_0(x) \tag{145}$$

The last job is to determine $f(x)$ for $x < 0$. Using that $S(0,t) = 0$ for any t we can define $f(x)$ for negative x as

$$f(x) = -g(-x) = -\frac{1}{2} S_0(|x|), \qquad x < 0 \tag{146}$$

Now $f(x)$ is a smooth function.

1.23 m/m01m2 T

When particle cross the point $x = 0$ with very small energy time increases logarithmically. If the energy of particle is E then using energy conservation low (it is violated only very slightly and during one period we can use it without changing E)

$$E = \frac{m\dot{x}^2}{2} - ax^2 + bx^4. \tag{147}$$

For small x

$$\dot{x} = \frac{2}{m}\sqrt{E + ax^2} \tag{148}$$

and time of crossing is

$$T(x_0) = \int_0^{x_0} \frac{dy}{\dot{x}(y)} \sim \frac{2}{m\sqrt{a}} \log E \tag{149}$$

Since after $E < 0$ the period reduces twice because the path reduces twice we approximately have

$$E(i) \sim (i_0 - i)\alpha \tag{150}$$

for $i < i_0$ and

$$E(i) \sim \frac{1}{2}(i - i_0)\alpha \tag{151}$$

for $i > i_0$.

Thus for $i < i_0$

$$T \sim \frac{2}{m\sqrt{a}} \log(i - i_0) + const \tag{152}$$

and

$$T \sim \frac{1}{m\sqrt{a}} \log(i - i_0) + const \tag{153}$$

for $i > i_0$

1.24 m/m01m3 T

We will solve part b. and then will get result for a. substituting $\beta = 0$. Let us introduce variable $z = y + ix$. Then the equation of motion will be

$$\ddot{z} + (\beta - i\frac{qB}{m})\dot{z} = g \tag{154}$$

Initial conditions $z(t = 0) = 0$ and $\dot{z}(t = 0) = 0$. The solution is

$$z(t) = \frac{img}{qB + im\beta} t + \frac{m^2 g}{(qB + im\beta)^2}[1 - e^{(i\frac{qB}{m} - \beta)t}] \tag{155}$$

and

$$\dot{z}(t) = \frac{img}{qB + im\beta}[1 - e^{(i\frac{qB}{m} - \beta)t}] \tag{156}$$

a). $\beta = 0$ This motion is the simultaneous circling and shifting.

b). In the case $\beta \neq 0$ radius of circling decreases and the motion became shifting with final velocity $\frac{img}{qB+im\beta}$.

c). The drag from b. changes the value and direction of shifting. In the case of radiation the direction and velocity of shifting coincide with the results from a. The final velocity is just the velocity of shifting.

To get this result we add the term $\alpha q \frac{d^3z}{dt^3}$ (α-some real number)to the equation of motion. Then the velocity depends on the time as

$$\dot{z}(t) = -\frac{img}{qB} + v_0[1 - e^{\Omega t}] \tag{157}$$

We have no idea to specify v_0. More interesting to find Ω

$$\Omega = \frac{1 - \sqrt{1 - 4\alpha q(\frac{iqB}{m})}}{2\alpha q} \tag{158}$$

When $\alpha = 0$ we return to the a. case. When $\alpha \neq 0$ we would like to note that

$$\Re\Omega < 0 \tag{159}$$

Thus velocity (and radius of circling decreases in time). The imaginary part is different from a. case- the frequency of circling is not the same as without drag.

1.25 m/m02m1 T

a).$z = r\cot\alpha$

$$L = \frac{m}{2}[\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\phi}^2] - mgr\cot\alpha \tag{160}$$

Equation of motion

$$mr^2\dot{\phi} = M = const \tag{161}$$

and

$$\frac{m}{\sin^2\alpha}\ddot{r} = -mg\cot\alpha + \frac{M^2}{mr^3} \tag{162}$$

b).If $\ddot{r} = 0$ then

$$w^2 = \frac{g}{z_0}\cot^2\alpha \tag{163}$$

c).Expanding we have $\Omega^2 = 3w^2\sin\alpha$

1.26 m/m02m2 T

a). Let ρ be $\frac{m}{l}$ The Lagrangian is

$$L = \frac{1}{2}\int_0^L dtdx[\rho\dot{y}^2 - \tau y'^2 + M\dot{y}^2(x,t)\delta(x - L/2)] \tag{164}$$

Equation of motion is

$$\rho\ddot{y}(x,t) - \tau y''(x,t) + M\ddot{y}(L/2)\delta(x - L/2) = 0 \tag{165}$$

This equation could be separated for two

$$\ddot{y} - v^2 y'' = 0 \tag{166}$$

where $v^2 = \frac{\tau}{\rho}$ for $x \neq L/2$ and

$$M\ddot{y}(L/2, t) = \tau[y'(l/2 + 0, t) - y'(l/2 - 0, t)] \tag{167}$$

Boundary conditions for $x = 0$ and $x = L$: $y = 0$.

We would like also to note that frequency (and thus wavelangth) to the left of mass and to the right of mass should be the same for $y(x)$ be smooth for any moment t in the center $x = L/2$.

Anzatc: $(w^2(k) = v^2k^2)$ for $x < L/2$

$$y = A_k \sin(w(k)t + \phi) \sin(kx) \tag{168}$$

for $x > L/2$

$$y = A_k \sin(w(k)t + \phi) \sin(k(L - x)) \tag{169}$$

Now we preserve smoothness of $y(x)$. But there is also exception when $y(L/2) = 0$ for any t:

$$k = \frac{2\pi n}{L} \tag{170}$$

Then y could be

$$y = A_k \sin(w(k)t + \phi) \sin(kx) \tag{171}$$

for $x < L/2$ and

$$y = -A_k \sin(w(k)t + \phi) \sin(kx) \tag{172}$$

for $x > L/2$

Even in this case dependence on time (and n) from both sides should coincides to satisfy equation for $M\ddot{y} = ...$

Now the equation of motion for mass M yields

$$Mw^2 \sin(\frac{kL}{2}) = 2\tau k \cos(\frac{kL}{2}) \tag{173}$$

Let us denote $\frac{kL}{2}$ as ξ. Then equation is

$$\tan\xi = \frac{m}{\xi M} \tag{174}$$

In the exceptional case (k through n) the equation for mass M already satisfied. This is just static wave with knot in the point $L/2$.

b). We did this. But except this there are such a solutions when $M = \infty$.

c). At $M = 0$ we have equation $tg\xi = infty$ and $k = \frac{(2n+1)\pi}{L}$

Plus exception $k = \frac{(2n\pi}{L}$

At $M = \infty$ we have equation $tg\xi = 0$ and $k = \frac{2n\pi}{L}$ This is the same wavelength as exceptional mode has (it should be added in the case $M = \infty$) but the amplitudes A_k in the RHS and LHS of the string has the same or opposite value correspondingly.

d). The first for frequencies (starting from $M = \infty$): two symmetric and two antisymmetric relatively center ($x = L/2$). Thus for each mode in the LHS we have two modes for system at all. Two modes in the LHS

$$y = \sin(w(n)t + \phi) \sin(\frac{2\pi nx}{L}) \tag{175}$$

and $n = 1, 2$.

When M changing from ∞ to 0 antisymmetric doesn't change! The symmetric modes are " sliding along tgh graph" (see equation derived before). Their momentum k increase and for finite M they looks like:

$n = 1$-back of two-humped camel. Each hump increase and became from part of sin from 0 to π into part of sin from 0 to $\frac{3}{2}\pi$

The same happens with $n = 2$. But for $M \neq \infty$ the symmetric mode is not fourth any more- the zero mode at $M = \infty$ (just constant mode) start " sliding along tgh graph" and goes to "one-hamped camel". Each side of hump in the end of the story ($M = 0$) became to sin from 0 to $\pi/2$.

1.27 m/m02m3 T

a).

$$\dot{v} = g - \frac{k}{m}v^2 \tag{176}$$

Solution

$$v = \sqrt{\frac{mg}{k}} \tanh \sqrt{\frac{kg}{m}}t \tag{177}$$

b).

$$\dot{u} = \cos \lambda w v \tag{178}$$

Solution

$$u = \cos \lambda \frac{mw}{k} \log \cosh \sqrt{\frac{kg}{m}}t \tag{179}$$

c).

$$v \to \sqrt{\frac{mg}{k}} \tag{180}$$

$$u \to wt\sqrt{\frac{mg}{k}} \cos \lambda \tag{181}$$

1.28 m/m02j1 T

Let us the distance between the $y = 0$ plane and mass M be h. And the angle between the horizontal line and stick of length $2c$ be ϕ. Then the constraint for this system is

$$b^2 = (a - c\cos\phi)^2 + (h + c\sin\phi)^2 \tag{182}$$

Potential energy of this system is

$$U = -2Mg[h + c\sin\phi] \tag{183}$$

Equilibrium point is $dU = 0$ or (using constraint)

$$\cos\phi_0 = \frac{a}{c} \tag{184}$$

$$h_0 + c\sin\phi_0 = b \tag{185}$$

and also

$$\frac{d\phi}{dh}(h_0) = -\frac{1}{a} \tag{186}$$

Now second derivative of U with respect to h at the equilibrium point:

$$U'' = -2Mg[a\frac{d^2\phi}{dh^2} - \frac{c}{a^2}\sin\phi_0] \tag{187}$$

Let us find $\frac{d\phi^2}{dh^2}$. Using constraint one could simply get

$$0 = (1 + c\sin\phi)^2 + (c\cos\phi\frac{d\phi}{dh})'(c\sin\phi - a) + (x + c\sin\phi)[c\cos\phi\frac{d^2\phi}{dh^2} - c\sin\phi\left(\frac{d\phi}{dh}\right)^2] + (c\cos\phi\frac{d\phi}{dh})^2 \tag{188}$$

At the equilibrium point this yields

$$\frac{d^2\phi}{dh^2}(h_0) = \frac{h_0 c}{a^3 b}\sin\phi_0 \tag{189}$$

Eventually

$$U'' = \frac{2Mgc[b - h_0]}{ba^2} \tag{190}$$

Now we will find the kinetic term. Horizontal speed of mass $M/2$ particles is $2\dot{h}\sin\phi_0\frac{c}{a}$. Horizontal speed of mass M particle is zero.

Vertical speed of mass $M/2$ particles is $\dot{h}[1 + \frac{2c\cos\phi_0}{a}] = -\dot{h}$. Vertical speed of mass M particles is $\dot{h}$.

Kinetic energy is $T = M\dot{h}^2[1 + 2\frac{c^2}{a^2}\sin^2\phi_0]$. The square of the frequency is

$$w^2 = \frac{2gc[b - h_0]}{b[a^2 + 2(b - h_0)^2]} \tag{191}$$

Now using that $(b - h_0)^2 = c^2 - a^2$ we have

$$w^2 = \frac{2gc\sqrt{c^2 - a^2}}{b(2c^2 - a^2)} \tag{192}$$

1.29 m/m02j2 T

Let us remind the relevant formulas for particle orbiting the central attractive force $\sim \frac{1}{r^2}$.

1. Angular momentum $L = mr^2\dot{\phi}$ is conserved.
2. If the potential energy is given by $-\frac{\alpha}{r}$ and total energy is equal to $-E$, $E > 0$ then the particle orbits the curve

$$r(\phi) = \frac{p}{1 + \epsilon\cos(\phi)} \tag{193}$$

where

$$p = \frac{L^2}{\alpha m} \tag{194}$$

and

$$\epsilon^2 = 1 - \frac{2L^2E}{m\alpha^2} \tag{195}$$

Now if the energy less then zero the orbiting is finite. Otherwise it is infinite.

Let us denote the velocity of satellite just before "collision" with Mars v and the velocity of Mars was $V = \sqrt{\frac{\alpha}{mb}}$. After "collision" the velocity of particle is $\vec{u} = 2\vec{V} - \vec{v}$.

The energy after the "collision" is

$$E = \frac{mu^2}{2} - \frac{\alpha}{b} = \frac{mv^2}{2} + \frac{\alpha}{b} - 2m\vec{v}\vec{V} \tag{196}$$

If before the interaction velocity of the satellite and Mars was directed opposite to each other $E > 0$ and satellite will go to space infinity.

To discuss the case of general directions we will need useful formulas

$$E = \frac{\alpha}{a+b} \tag{197}$$

and

$$L^2 = \frac{2abm\alpha}{a+b} \tag{198}$$

Then the energy after colliding (in assumption that $\overrightarrow{V}$ and $\overrightarrow{v}$ were parallel to each other) is

$$E = -\frac{\alpha}{b}[\frac{a}{a+b} + 1 - 2\sqrt{\frac{2a}{a+b}}] \tag{199}$$

This energy could be negative. And the square of angilar moment is

$$L^2 = 4\alpha mb[1 - \sqrt{\frac{a}{2(a+b)}}]^2 \tag{200}$$

Eventually the largest distance is

$$\frac{4p}{1-\epsilon}\left(1 - \sqrt{\frac{a}{2(a+b)}}\right)^2 \tag{201}$$

where

$$\epsilon = \sqrt{1 - 8[2\sqrt{\frac{2a}{a+b}} - 1 - \frac{1}{a+b}]\left(1 - \sqrt{\frac{a}{2(a+b)}}\right)^2} \tag{202}$$

1.30 m/m02j3 T

a).$T = k\delta x = kl\frac{\delta l}{l}$. Now $\frac{\delta l}{l}$ should be substituted by $\frac{\delta L}{L}$. Eventually we have $\frac{kl}{L}$.

b).

$$T(x) = kl(\frac{ds}{dx} - 1) \tag{203}$$

and

$$\frac{m}{l}g = -\frac{dT}{dx} \tag{204}$$

Solving this with boundary conditions: $s(0) = 0$ and $T(l) = Mg$ we have

$$S_0(x) = (1 + \frac{(m+M)g}{kl})x - \frac{mgx^2}{2kl^2} \tag{205}$$

and $S_0(l) = l + \frac{(m+2M)g}{2k}$. c). Now it is useful to introduce new variable $D(x,t) = S(x,t) - S_0(x)$. Wave equation for $D(x,t)$ contains $l\sqrt{\frac{k}{m}}$ as the speed of waves. Plus boundary conditions $D(t,0) = 0$ and $M\ddot{D}(t,l) = -klD'(t,l)$.

Using first constraint we construct our solution in the form

$$e^{iwt}\sin(px) \tag{206}$$

Then the other constraint yields

$$\tan(pl) = \frac{m}{Mpl} \tag{207}$$

and also $w = pl\sqrt{\frac{k}{m}}$ d). $M = 0$ yields $p = \frac{\pi}{2l}$. "A half of the wave".

$m = 0$ naively yields $p = 0$-no oscillations at all. But this is too naive. We will present correct answer after the intermediate case $m << M$.

In the case $m << M$ we can expand tan and get $p = \frac{1}{l}\sqrt{\frac{m}{M}}$ (here we are interested in the low frequency). As was expected in this case $w = \sqrt{\frac{k}{M}}$. The last step is to normalize $D(x,t)$ in the way that amplitude of the oscillations of the mass M become finite. Or

$$D(x,t) = e^{i\sqrt{\frac{k}{M}}t}\frac{x}{l} \tag{208}$$

Now we can simply take the limit $m = 0$ and get the usual result for pendulum handing on the massless string .

1.31 m/m03m1 T

a). Third Newtons low for orbit of radius $r > R$

$$mw^2 r = \frac{mM_s G}{r^2} + \frac{mM_e G}{(r-R)^2} \tag{209}$$

where w is the frequency of Earth $w^2 = \frac{M_s G}{R^3}$. This equation could be rewritten in the way $r = \xi R$

$$1 = \frac{1}{\xi^3} + \frac{M_e}{M_s \xi(\xi-1)^2} \tag{210}$$

Obviously it has one solution for $xi > 1$. Really starting at $\xi = \infty$ and when going to $\xi = 1$ RHS increases from 0 to infinity. Thus it crosses 1 at explicitly one value of x.

b). Taking $\frac{M_e}{M_s} = \epsilon$ as a small parameter we understand that xi should goes to 1 when ϵ goes to 1. Or, in other words, when ξ is finite (approximately 1) quantity $\xi - 1$ is small. We would like to rewrite the previous equation in the form

$$\epsilon = \frac{(x-1)^3(x^2+x+1)}{x^2} \tag{211}$$

According to the previous speculations we could find the answer in the leading order by treating $\xi = 1$ and looking for $(\xi - 1)$:

$$\xi = 1 + \left(\frac{\epsilon}{3}\right)^{1/3} \tag{212}$$

Numerically (if $\epsilon = \beta$) $\xi = 1 + 10^{-2}$ and $(r - R) = 110^6$km.

c).Using angular momentum conservation we could find square of angular frequency of small oscillations

$$\Omega^2 = 3w^2 - 2[\frac{M_s G}{r^3} + \frac{M_s G}{(r-R)^3}] \tag{213}$$

We are interesting whether $\Omega^2 > 0$ or not. Rewriting this we will get

$$3 - \frac{2}{\xi^3} - \frac{2\epsilon}{(\xi-1)^3} \tag{214}$$

Using our equation for ξ through ϵ we can reexpressed the result only through xi (equivalent inequality)

$$3 - 2[1 + \xi^{-1} + \xi^{-2} + \xi^{-3}] > 0 \tag{215}$$

Thus we have that orbiting is stable for large xi (when particle is far from the Earth) and unstable otherwise. The infernal point is when

$$\xi^4 - 3\xi^2 + 2 = 0 \tag{216}$$

In our case, when $\xi = 1 + \left(\frac{\epsilon}{3}\right)^{1/3}$ we have

$$3 - 2[1 + \xi^{-1} + \xi^{-2} + \xi^{-3}] = -5 + O(\epsilon^{1/3}) < 0 \tag{217}$$

The orbit is unstable.

1.32 m/m03m2 T

a). Second Newton's low

$$M\ddot{\overrightarrow{R}} = m\overrightarrow{g}\sin\theta + \overrightarrow{f} \tag{218}$$

(our $\overrightarrow{g}$ has the value of usual g but directed "downhill")

and the same low for angular momentum

$$I\dot{w} = a[\overrightarrow{n} \times \overrightarrow{f}] \tag{219}$$

with without slipping constraint

$$\frac{d}{dt}\dot{R} = \frac{d}{dt}a[\overrightarrow{w} \times \overrightarrow{n}] \tag{220}$$

yield

$$\overrightarrow{f}[1 + \frac{Ma^2}{I}] = -m\overrightarrow{g}\sin\theta \tag{221}$$

Hence effective force (RHS in the Newton's low) is

$$m\overrightarrow{g}\sin\theta[1 - \frac{1}{1 + \frac{Ma^2}{I}}] = \frac{5}{7}\overrightarrow{g}\sin\theta \tag{222}$$

in the case $I = \frac{2}{5}Ma^2$

b). Once again second Newton's low

$$M\ddot{\overrightarrow{R}} = \overrightarrow{f} \tag{223}$$

plus low for angular momentum

$$I\dot{w} = a[\overrightarrow{n} \times \overrightarrow{f}] \tag{224}$$

plus without slipping condition

$$\dot{R} = a[\overrightarrow{w} \times \overrightarrow{n}] + [\overrightarrow{\Omega} \times \overrightarrow{R}] \tag{225}$$

($R = 0$-center of rotation) yield

$$\overrightarrow{f} = \frac{M[\overrightarrow{\Omega} \times \dot{\overrightarrow{R}}]}{1 + \frac{Ma^2}{I}} \tag{226}$$

and

$$\ddot{R} - \frac{[\Omega \times \dot{R}]}{1 + \frac{Ma^2}{I}} = 0 \tag{227}$$

Let us denote λ as

$$\lambda = \frac{}{1 + \frac{Ma^2}{I}} = \frac{2}{7} \tag{228}$$

Then introducing complex cartesian coordinates on the plane $r = x + iy$ and $v = \dot{x} + i\dot{y}$ the equation became

$$\ddot{v} + (\lambda\Omega)^2 v = 0 \tag{229}$$

with solution -linear superposition of

$$Ae^{i\lambda\Omega t} + Be^{-i\lambda\Omega t} \tag{230}$$

Obviously this is orbiting (arbitrary point as center) with frequency

$$\lambda\Omega \tag{231}$$

1.33 m/m03m3 T

a). We will assign the gas such quantities as field of pressure, field of density and field of velocity. We also assume that our process is adiabatic, or

$$P\rho^{-\gamma} = const \tag{232}$$

Let us consider small (pointlike) volume of air. Its acceleration $\dot{v}$ (as a hole object is)

$$\dot{v} = \frac{\partial P}{\rho \partial x} \tag{233}$$

We can also connect velocity of air with gradient of density using conservation of mass equation

$$\frac{d\rho}{dt} + grad(\rho v) = 0 \tag{234}$$

More convenient for us form of this equation is

$$\frac{\partial\rho}{\partial t} + \rho grad(v) = 0 \tag{235}$$

After all we have

$$\ddot{\log}\rho + \partial(\rho^{-1}\partial P) \tag{236}$$

Using first equation one will have ($\phi = \rho^{\gamma-1}$)

$$\ddot{\log}\phi + \gamma P_0 \rho_0^{-\gamma}\Delta\phi = 0 \tag{237}$$

Expanding around $\rho = rho_0$ one will get usual wave equation with

$$v_s^2 = \frac{\gamma P_0}{\rho_0} \tag{238}$$

b). Obviously at the boundary velocity should never be nonzero (vacuum can not appear and particles can not penetrate through the wall). Thus $\partial P = 0$ at the boundary.

Eigenmodes are

$$\delta\rho = A_{n_x,n_y,n_z}\rho_0 e^{\pm iw(n)t}\cos(\frac{x\pi n_x}{L})\cos(\frac{y\pi n_y}{L})\cos(\frac{z\pi n_z}{L}) \tag{239}$$

Here $w^2(n) = \frac{v_s^2\pi^2 n^2}{L^2}$

We should note that wave equation is the consequence of all equation we wrote. But there is also additional condition: number of particles should not change with time, or

$$\int d^3x\delta\rho = 0 \tag{240}$$

This condition kills only the mode with $w = 0$ (without spatial dependence).

c). If the pressure in the cube is $P(t)$ then density there is $\frac{P^{1/\gamma}}{P_0^{1/\gamma}}\rho_0$ and amount of gas (mass) which has to leave the cube is (in linear in δP approximation)

$$\delta m = -\frac{L^3\rho_0\delta P}{\gamma P_0} \tag{241}$$

This mass has volume (inside the tube) equal to

$$Sx\rho_0 = \delta m \tag{242}$$

Here x is a length of the tube of air from cube. We should not worry about negative x- we could add to this volume arbitrary volume from tube to make it positive. Now $\ddot{x} = -\frac{L^3\delta\ddot{P}}{S\gamma P_0}$. But from the Newtons low

$$m\ddot{x} = S(P - P_{atm}) \tag{243}$$

Here $m = \rho_0 Sl$-full mass of air inside the tube and P_{atm} is the pressure outside the tube. We will introduce now new variable $p(t) = P(t) - P_{atm}$. Then equation of motion is

$$\ddot{p} + \Omega^2 p = 0 \tag{244}$$

and

$$\Omega^2 = \frac{v_s^2 S}{L^3 l} \tag{245}$$

2 Electrodynamics

2.1 e/e98j1 V

From the equations of electrodynamics in media

$$\nabla \mathbf{D} = 4\pi\rho \tag{246}$$

$$\nabla \mathbf{B} = 0 \tag{247}$$

$$\nabla \times \mathbf{H} = \tfrac{1}{c}(4\pi\mathbf{j} + \dot{\mathbf{D}}) \tag{248}$$

$$\nabla \times \mathbf{E} = -\tfrac{1}{c}\dot{\mathbf{B}} \tag{249}$$

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{250}$$

$$\mathbf{B} = \mu\mathbf{H} \tag{251}$$

$$\mathbf{j} = \sigma\mathbf{E} \tag{252}$$

for the case of the plane wave of the form $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}e^{i\mathbf{kr}-i\omega t}$, $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}e^{i\mathbf{kr}-i\omega t}$, unit magnetic permeability $\mu = 1$ and zero external sources $\rho = 0$ we get

$$\mathbf{kD} = 0 \tag{253}$$

$$\mathbf{kB} = 0 \tag{254}$$

$$i\mathbf{k} \times \mathbf{H} = \tfrac{1}{c}(-i\omega\varepsilon\mathbf{E} + 4\pi\sigma\mathbf{E}) \tag{255}$$

$$i\mathbf{k} \times \mathbf{E} = -\tfrac{1}{c}(-i\omega\mathbf{H}) \tag{256}$$

and the immediate consequence

$$\mathbf{k}^2 = \frac{\omega^2}{c^2}\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right) \equiv n^2\frac{\omega^2}{c^2}, \qquad n = \sqrt{\varepsilon + i\frac{4\pi\sigma}{\omega}}. \tag{257}$$

For given numbers one computes

$$k = \sqrt{1/3e10^2 * (2.5e9^2 * 50 + i * 4 * pi * 2e10 * 2.5e9)} = 0.75 + 0.46i \tag{258}$$

and $l = 2.2cm$.

2.2 e/e98j2 V

See problem e00j3 for derivation of the intensity of radiation of an accelerating charge. The condition $\omega \ll \frac{c}{l}$ implies that the size of radiating system is much less than the wave length, therefore the approximation e00j3 is valid:

$$I_e = \frac{2}{3c^3}\ddot{d}^2 = \frac{2}{3c^3}(\lambda l \frac{1}{2}\omega^2 l)^2 \tag{259}$$

The magnetic dipole moment is suppressed by the ration v/c, therefore the intensity of its radiation will be

$$I_m \propto I_e \left(\frac{\omega l}{c}\right)^2 \tag{260}$$

2.3 e/e98j3 V

Just near the surface

$$E = E_0 + \frac{1}{R^3}(3|p|\cos\theta\overrightarrow{n} - \overrightarrow{p}) \tag{261}$$

Since inside $E = 0$ we obtain charge density

$$\sigma = E_0 \frac{3}{4\pi} \cos\theta \tag{262}$$

and the field which acts on a charge but not created by this charge is

$$E_{ext} = E_0 \frac{3}{2} \cos\theta \tag{263}$$

Therefore the force

$$F = \int 2\pi d \cos\theta \cos\theta E_0 \frac{3}{2} \cos\theta E_0 \frac{3}{4\pi} \cos\theta = 3E_0^2 R^2 \tag{264}$$

2.4 e/e98m1 V

From

$$A_\mu = \frac{1}{c} \frac{j_\mu(t - r/c)}{r} \tag{265}$$

integrating the current in the limits $\pm\sqrt{(tc)^2 - x^2}$ we get

$$A_z = \frac{\alpha}{c} \left[t \log \frac{\sqrt{(tc)^2 - x^2} + tc}{-\sqrt{(tc)^2 - x^2} + tc} - \frac{2}{c}\sqrt{(tc)^2 - x^2} \right] \tag{266}$$

and

$$H_\phi = -\partial_r A_z = -\frac{2\alpha}{c^2 x}\sqrt{(tc)^2 - x^2} \tag{267}$$

and

$$E_z = -\partial_t A_z = -\frac{\alpha}{c} \log \frac{\sqrt{(tc)^2 - x^2} + tc}{-\sqrt{(tc)^2 - x^2} + tc} \tag{268}$$

The limiting cases are read from the expressions above in an evident way.

2.5 e/e98m2 V

Since the magnet will attract 'magnetic charges' to the boundary of the media with infinite magnetic permeability in such a way to cancel the magnetic field, they will create the field $2\pi\sigma$, in this field the charge σA is located, therefore the force

$$F = 2\pi\sigma\sigma A \tag{269}$$

(in the problem M is given. It is σ).

2.6 e/e98m3 V

a. The field of dipole is

$$H = \frac{(\overrightarrow{p}\,\overrightarrow{r})\overrightarrow{r} - \overrightarrow{p}r^2}{r^5} \tag{270}$$

From the field at the equator, neglecting $\theta = 11^0$, we get

$$p = HR^3 = 0.5 \cdot (6.4 \cdot 10^8)^3 = 0.13 \cdot 10^{27}\,\text{gauss units} \tag{271}$$

b. See e00j3 for radiation of accelerating charge.

$$P = \frac{2}{3}\frac{\ddot{p}^2}{c^3} = \frac{2}{3}\frac{(p\omega^2 \sin\theta)^2}{c^3} = 350\,\text{gauss units} = 3.5 \cdot 10^{-5}\text{Wt} \tag{272}$$

c. In plasma the dispersion relation for the plane wave is

$$n = \sqrt{1 - \frac{4\pi n q^2}{m\omega^2}} \tag{273}$$

(it is immediate sequence of Maxwell equations together with action of field on the charges with the plane wave anzatz)

$$\omega_c = \sqrt{\frac{4\pi n q^2}{m}} \approx 4 \cdot 10^3 Hz \tag{274}$$

Since $\omega_{earth} \ll \omega_{crit}$ the refraction index is purely imaginary, and the characteristic length of damping of the signal is $l = \frac{c}{\omega_{crit}} \approx 7 \cdot 10^2$km, that is enormously less than space scales in the solar system. Therefore, the signal is undetectable.

2.7 e/e99j1 V

a. The force on dipole is

$$\overrightarrow{F} = (\overrightarrow{p}\,\overrightarrow{\nabla})\overrightarrow{E} = \alpha(\overrightarrow{E}\,\overrightarrow{\nabla})\overrightarrow{E} \tag{275}$$

If $E(x,t) = e(x)\cos(\omega t + \phi(x))$, then

$$\left\langle \overrightarrow{F} \right\rangle = \frac{\alpha}{2}(\overrightarrow{e}\,\overrightarrow{\nabla})\overrightarrow{e} \tag{276}$$

The force is directed towards region with stronger field.

b. For the wave in complex notations , when $E_x = \Re e^{ikx - i\omega t}$ and $E_y = \mathrm{Im}\, e^{ikx - i\omega t}$ imaginary part in the polarization will effectively mean, that the $p = \alpha E$ is directed not along E but at an angle $\tan\theta = \frac{a'}{a''}$, then there is non zero force acting on this rotation dipole from the magnetic field B. Therefore,

$$F = \frac{\omega}{c}\alpha''|E|^2 \tag{277}$$

c. From

$$m\ddot{x} + \gamma m\dot{x} + m\omega_0^2 x = qE \tag{278}$$

for $E = Ee^{i\omega t}$ one gets

$$x = \frac{qE}{m}\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} \tag{279}$$

the real part

$$\alpha' = \frac{q}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{280}$$

has maximum at

$$\omega = \sqrt{\omega_0^2 - \gamma\omega_0} \tag{281}$$

2.8 e/e99j2 V

Stress energy tensor $T = -2\frac{\delta S}{\delta g}$ from the lagrangian $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ is

$$T^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\sigma}F^{\nu}_{\sigma} + \frac{1}{4}g^{\mu\nu}F_{lm}F^{lm}\right) \tag{282}$$

It's space components are the flow of momentum (3x3 pressure-tension tensor):

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left(-E_\alpha E_\beta - H_\alpha H_\beta + \frac{1}{2}\delta_{\alpha\beta}(E^2 + H^2)\right) \tag{283}$$

Due to the interference between the external field and the field from the charge there is flow of momentum throw the spherical surface separating the charge

$$F_\alpha = \int_{S^2} \sigma_{\alpha\beta} dn_\beta = \frac{1}{4\pi}\int dn_\beta \left((-(p_\alpha + n_\alpha)(p_\beta + n_\beta) + \frac{1}{2}\delta_{\alpha\beta}2(pn))\right) = qp_a \tag{284}$$

where the integral is taken over S^2 around the charge, with $p_a = E^{ext}_a$ and $n_a = E^q_a$. Therefore, the momentum flows away from the charge in accordance to the force acting on it.

2.9 e/e99j3 V

Due to rotation the star looses energy with power

$$\frac{d}{dt}\left(\frac{J\omega^2}{2}\right) = -P = -\frac{2}{3}\frac{\mu^2\omega^4}{c^3} \tag{285}$$

from $J\omega\dot{\omega} = -P$ and $\dot{T} = -\frac{2\pi}{\omega^2}\dot{\omega}$ we get

$$\mu = \sqrt{\frac{3}{2}\frac{\dot{T}T}{4\pi^2}c^3 J}, \tag{286}$$

where $J = \frac{2}{5}mR^2$, and since $B_{max} = 2\frac{\mu}{R^3}$ we get

$$B_{max} = \sqrt{\frac{3}{5\pi^2}\frac{\dot{T}Tmc^3}{R^4}} = 0.16 \cdot 10^{16}\text{gauss} \tag{287}$$

And

$$E^{QED}_{crit} = \frac{m^2c^3}{e\hbar} = 0.5 \cdot 10^{14} \tag{288}$$

thus B_{max} is stronger in ≈ 30 times.

2.10 e/e99m1 V

Since the sphere contains charge,due to rotation it has the magnetic moment. For the classical sphere the relation is

$$\mu = \frac{Q}{2Mc}L \tag{289}$$

where L is the mechanical momentum. In the magnetic field the sphere precess

$$J\dot{\omega} = [\mu\omega \times B] \tag{290}$$

with the frequency $\Omega = \frac{\mu B}{J} = \frac{QBJ\omega}{2mcJ} = \frac{Q}{M}\frac{\omega B}{2c}$, which leads to

$$\frac{Q}{M} = \frac{4\pi^2c^2}{\omega B\lambda} \tag{291}$$

The polarization of this dipole radiation is circular.

2.11 e/e99m2 V

Let the radius of the wheel is ρ. The resistance of a separate spike is $r = \lambda\rho$, of a separate segment on a circle is $R = \lambda\rho\frac{2\pi}{5}$. The resistance between the center and the ending of one of spike is

$$r_{eff} = r + r/4 + 2R = \frac{5}{4}r + \frac{1}{2}R \tag{292}$$

When the spike passes the magnetic field, it creates induced electrical field, which causes current, which dissipates.

$$U = -\dot{\Phi} = \frac{1}{2}\omega B\rho^2 \tag{293}$$

The rate of loosing of the kinetic energy is

$$\frac{d}{dt}\frac{J\omega^2}{2} = \frac{U^2}{r_{eff}} = (\frac{1}{2}B\rho^2)^2\frac{1}{r_{eff}}\omega^2 \tag{294}$$

therefore the change of velocity due to passing through the wedge of one of the spikes is

$$\omega(t) = \omega(0)e^{-\frac{(B\rho^2)^2}{4Jr_{eff}}t} \tag{295}$$

Since there is no picture, not loosing generality, let us assume that the angle of the wedge θ is less than $\frac{2\pi}{5}$

The angle depends on time like

$$\phi_i(t) = \phi_i^0\frac{\omega_i}{\alpha}(1 - e^{-\alpha t}), \tag{296}$$

where $\alpha = \frac{(B\rho^2)^2}{4Jr_{eff}}$. If $\phi(\infty)$ less then than the angle of the wedge, the wheel will stop, having still the same spike in the magnetic field. If it is greater, the wheel will leave with this spike the region of magnetic field with the angular velocity ω_{i+1}

$$\omega_{i+1} = \omega_i - \alpha\theta \tag{297}$$

Then it will rotate freely until the next spike enter the magnetic field, an the process of slowing down will repeat with the same functional form as (295).

2.12 e/e99m3 V

By definition of the electric permittivity the following relation holds for a flat layer in the perpendicular electric field

$$E_{tot} = E_{ext} + E_{pol} \tag{298}$$

$$E_{pol} = -4\pi\alpha E_{tot}, \qquad \Rightarrow E_{tot} = \frac{E_{ext}}{1+4\pi\alpha}, \qquad E_{tot} = \frac{E_{ext}}{\varepsilon}, \qquad \varepsilon = 1 + 4\pi\alpha \tag{299}$$

Then, αE_{tot} has a sense of the specific electric dipole moment per volume. In the 2D picture this distributed electric dipole moment creates homogenous field, and therefore the problem is solved exactly (symmetry of the disk and Gauss theorem is exploited).

$$E_{pol} = -\frac{\pi}{2\pi}4\pi\alpha E_{tot} = -2\pi\alpha E_{tot} \tag{300}$$

and therefore, inside the disk

$$E_{tot} = \frac{1}{1+2\pi\alpha}E_{ext} = \frac{2}{1+\varepsilon}E_{ext} \tag{301}$$

Outside the disk the field is the superposition of the dipole field and the external field

$$\vec{E} = \vec{E}_{ext} + \vec{E}_{dip} = \vec{E}_{ext} + \frac{3(\vec{p}\,\vec{r})\vec{r} - \vec{p}\,\vec{r}}{r^5} \tag{302}$$

where

$$\vec{p} = \pi R^2\frac{\varepsilon-1}{4\pi}\frac{2}{1+\varepsilon}\vec{E}_{ext} = \frac{1}{2}\frac{\varepsilon-1}{\varepsilon+1}\vec{E}_{ext} \tag{303}$$

2.13 e/e00j1 V

One can solve the Laplace equation with the given boundary conditions on the circles in 2D, using the methods of conformal mappings, or one can just note, that two opposite charges with 2D logarithmic potential have circles as the curves of constant potential. Let the potential on the left circle be $-\phi_0$, on the right ϕ_0 and we will choose such charges and positions for them to create such potential on given circles. Let $\overrightarrow{p} = (\pm p, 0)$ be position of the charges.

$$\phi(r) = 2q(\log|\overrightarrow{r} - \overrightarrow{p}| + \log|\overrightarrow{r} + \overrightarrow{p}|) \tag{304}$$

The equation of curve of constant potential const $= \alpha = e^{\frac{\phi}{2q}}$ is

$$y^2 + \left(x + p\frac{\alpha^2+1}{\alpha^2-1}\right)^2 = p^2\frac{4\alpha^2}{(\alpha^2-1)^2} \tag{305}$$

from which the position of its center

$$b = p\frac{\alpha^2+1}{\alpha^2-1} \tag{306}$$

and its radius is

$$R = p\frac{2\alpha}{\alpha^2-1}, \tag{307}$$

Solving for p, α in terms of R, b we get

$$p^2 = b^2 - R^2; \qquad \alpha = \frac{1}{R}(\sqrt{b^2-R^2}+b) \tag{308}$$

and the capacity is

$$C = \frac{q}{2\phi_0} = \frac{1}{4}\frac{1}{\log\left(\frac{\sqrt{b^2-R^2}+b}{R}\right)} \tag{309}$$

At large b it agrees with the naive estimation (if one neglects the displacement of the charges on one wire due to the field of the other wire)

$$C_{naive} = \frac{1}{4}\frac{1}{\log(\frac{2b-R}{R})} \tag{310}$$

2.14 e/e00j2 V

The gradient of magnetic field causes the appearance of the circular electrical field, which cause the current, which creates magnetic moment, which interact with the the gradient of the magnetic field and slows the circle.

$$E = \frac{a}{2c}\partial_z Bv; \qquad I = \pi b^2\sigma E = \frac{\pi^2 ab^2\sigma}{2c}v\partial_z B; \qquad \mu = \frac{1}{c}IS \tag{311}$$

$$m\dot{v} = F = -\mu\partial_z B = \frac{\pi^3 a^2 b^2\sigma}{2c^2}(\partial_z B)^2 v \tag{312}$$

The gradient $\partial_z B$ for the solenoid is obviously proportional to the field from the one, the nearest circular loop

$$\partial_z B = n\frac{2\pi Ir}{c}\frac{r}{(r^2+z^2)^{3/2}} \tag{313}$$

where r is the radius of the solenoid and n is the density of loops. Rewriting the differential equation with respect to z from t with $\dot{v} = v'v$ and integrating we get

$$\Delta(mv) = \frac{\pi^3 a^2 b^2\sigma}{2c^2}\left(\frac{2\pi nIr^2}{c}\right)^2\int_0^\infty \frac{dx}{(x^2+r^2)^3} = \frac{\pi^3 a^2 b^2\sigma}{2c^2}\left(\frac{2\pi nIr^2}{c}\right)^2\frac{1}{r^5}\frac{3\pi}{16} \tag{314}$$

2.15 e/e00j3 V

From the Lagrangian

$$-\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}-\frac{1}{c^2}A_\mu j^\mu \tag{315}$$

follows the equation of motion

$$\partial_\mu F^{\mu\nu}=\frac{4\pi}{c}j^\nu, \tag{316}$$

which in the Lorentz gauge $\partial_\mu A^\mu=0$ yields $(\partial_t^2-\partial_r^2)A^i=\frac{4\pi}{c}j^i$ with the solution

$$A^i(t,y)=\frac{1}{c}\int\frac{j(t-|y-x|/c,x)}{|y-x|}d^3x \tag{317}$$

Further than in the wavelength region the wave can be considered to be plane with

$$\overrightarrow{H}=\frac{[\overrightarrow{n}\times\overrightarrow{j}]}{c^2R} \tag{318}$$

and $E=\frac{1}{c}[\overrightarrow{H}\times\overrightarrow{n}]$, and the energy flow $\overrightarrow{P}=\frac{c}{4\pi}H^2\overrightarrow{n}$. At some point there are two contributions to the total field from two different sources. They are averaged over the time with $\left\langle[\cos(\omega t)+\cos(\omega t+\phi)]^2\right\rangle=1+\cos(\phi)$ The phase shift is contributed from different phase source supply and from different path of propagation $\phi=\alpha+\frac{\omega\Delta}{c}sin\theta\sin\phi$, where $\alpha=\pi$ in the current problem. Thus, the answer is

$$\frac{dP}{d\Omega}=\frac{1}{4\pi c^3}\sin^2\theta\omega^2I^2\left(\frac{d}{2}\right)^2\left(1+\cos(\alpha+\frac{\omega\Delta}{c}\sin\theta\sin\phi)\right) \tag{319}$$

2.16 e/e00m1 V

As in e98j1 we get

$$k=\sqrt{\frac{\omega^2\varepsilon}{c^2}+\frac{i\omega 4\pi\sigma}{c^2}} \tag{320}$$

when $\frac{4\pi\sigma}{\varepsilon\omega}\gg 1$ we get

$$k=\sqrt{i}\sqrt{\frac{4\pi\sigma\omega}{c^2}} \tag{321}$$

by boundary conditions we get that at the surface y,z with the wave going in the plane y,x

$$k_{y0}=k_{y1} \tag{322}$$

and

$$k_{xi}^2=k_{x0}^2+k_0^2(n_i^2-1) \tag{323}$$

then, since transversal component of B and E is continuous we get

$$r=\left|\frac{n_x^l-n_x^r}{n_x^l+n_x^r}\right|^2, \tag{324}$$

where $n_x^{TE}=\frac{k_x}{k_0}$ for transversely polarized electric wave, and $n_x^{TM}=\frac{k_x}{n^2k_0}$ for transversely polarized magnetic wave. By Kirchhoff's law

$$t=1-r, \tag{325}$$

Then, in the limit $A=\frac{4\pi\sigma}{\varepsilon\omega}\to\infty$ we get

$$t^{TE}=2\sqrt{2}\frac{\cos\theta}{\sqrt{A}}, \tag{326}$$

$$t^{TM}=2\sqrt{2}\frac{1}{\cos\theta\sqrt{A}}, \tag{327}$$

Thus, at $\theta\to\pi/2$, the polarization become transversely magnetic.

2.17 e/e00m2 V

a. Since there is image charge inside the plane,

$$m\ddot{x} = -\frac{q^2}{(2x)^2} \tag{328}$$

and integrating equations of motion

$$T = \sqrt{\frac{2mx_0^3}{q^2}} \int_0^1 \frac{dt}{\sqrt{1 - 1/t}} \tag{329}$$

b. Radiated power

$$P = \frac{2}{3}\frac{(q\ddot{x})^2}{c^3} = \frac{1}{6}\frac{q^6}{m^2 x^4 c^3} \tag{330}$$

c. By placing charges e' at the opposite point and e'' at the same point, and using $E_{1t} = E_{2t}, \qquad D_{1n} = D_{2n}$ we get

$$e + e' = e''/\varepsilon, \qquad e - e' = e'' \tag{331}$$

then, $e' = -\frac{\varepsilon - 1}{\varepsilon + 1}$ and the force

$$F = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q^2}{(2x)^2} \tag{332}$$

2.18 e/e00m3 V

a. From Om's equations

$$V_0 = -L_{11}\dot{I}_1 - L_{12}\dot{I}_2 \tag{333}$$

$$RI_2 = -L_{12}\dot{I}_2 - L_{22}\dot{I}_2 \tag{334}$$

we get

$$I_2 = V_0 \left(i\omega \left(L_{12} - L_{22}\frac{L_{11}}{L_{12}} \right) + R\frac{L_{11}}{L_{12}} \right)^{-1} \tag{335}$$

and the dissipated power is

$$\langle P \rangle = \frac{1}{2}|I_2|^2 R \tag{336}$$

b. From definition of inductances $\Phi_i = L_{ij} I_j$ and simple expression for magnetic field in the solenoid we get

$$B_1 = \frac{\mu_0 N_1 I_1}{L} \tag{337}$$

and

$$L_{11} = \frac{\mu_0 N_1 N_1 S_1}{L}, \tag{338}$$

$$L_{21} = \frac{\mu_0 N_1 N_2 S_2}{L}. \tag{339}$$

Analogously,

$$B_2 = \frac{\mu_0 N_2 I_2}{L} \tag{340}$$

$$L_{12} = \frac{\mu_0 N_1 N_2 S_2}{L}, \tag{341}$$

$$L_{22} = \frac{\mu_0 N_2 N_2 S_2}{L}. \tag{342}$$

2.19 e/e01j1 V

By conformal map of the unit disk to the half plane

$$w(z) = \frac{z-1}{z+1} \tag{343}$$

we get that distance between cuts will be

$$d' = 2R\tan\frac{\phi}{2} \tag{344}$$

where 2ϕ is the angle between them on the disk, and the size of cuts will be

$$\delta' = \delta(\tan\frac{\phi}{2})' = \frac{1}{2}\delta\frac{1}{\cos^2\frac{\phi}{2}} \tag{345}$$

At the plane we find that the resistance is

$$R = \frac{\Delta U}{I} = \frac{2E_{\delta'}\delta'\log\frac{d'}{\delta'}}{\pi\delta'\sigma a E_{\delta'}} = 2\frac{\log\frac{d'}{\delta'}}{\pi\sigma a} = \frac{2}{\pi\sigma a}\log\frac{d}{\delta} \tag{346}$$

2.20 e/e01j2 V

Since

$$\left.\frac{dp}{dt}\right|_r = \omega p = \frac{qvB}{c} \tag{347}$$

we get

$$p = \frac{qRB}{c} \tag{348}$$

Since

$$\left.\frac{dp}{dt}\right|_{tan} = qE, \tag{349}$$

and

$$E = \frac{1}{2\pi R}\frac{1}{c}\pi R^2\dot{B}_{av} = \frac{R}{2c}\dot{B}_{av} \tag{350}$$

we get

$$B_{av} = 2B \tag{351}$$

For ultra-relativistic electrons

$$\dot{p} \approx \dot{\varepsilon} \tag{352}$$

and

$$cq\frac{R}{2c}\dot{B}_{av} = \dot{\varepsilon} = \frac{2}{3c^3}q^2w^2 = \frac{2}{3c^3}q^2\left(\frac{\omega v}{1-v^2}\right)^2. \tag{353}$$

Thus,

$$\varepsilon_{max} = m\left(\frac{3R^3\dot{B}_{av}}{4cq}\right)^{\frac{1}{4}} \tag{354}$$

2.21 e/e01j3 V

In the media, where $k(n) = k_0 n = \frac{\omega n}{c}$,

$$\tilde{f}(z,t) = \int d\omega e^{i(\omega_0 + b(\omega-\omega_0)\frac{z}{c} - \omega t)} f_\omega = \int d\omega e^{i(\omega_0 + b(\omega-\omega_0)\frac{z}{c} - \omega t)} \frac{1}{2\pi} \int e^{-i\omega x} f(x) = f(b\frac{z}{c} - t) \tag{355}$$

where $f(\xi) = \int f_\omega d\omega e^{i\omega\xi}$, and $\frac{b}{c} = \frac{dk}{d\omega}$ is just the inverse 'group speed'. In accordance to its name, the pulse propagates in the media with the velocity $v_g = \frac{d\omega}{dk}$. After the media, the pulse is the same, shifted in time

$$\tilde{f_{aft}}(z,t) = f(\frac{z}{c} - t + \tau) \tag{356}$$

where $\tau = \frac{(b-1)a}{c}$

2.22 e/e01m1 T

a).

$$E(r) = \frac{V_0}{\phi_0 r} \tag{357}$$

And energy

$$\mathcal{E} = d\int_b^{b+c} \frac{1}{2} E^2 \phi_0 r = \frac{V_0^2 d}{2\phi_0} \log(1 + \frac{c}{b}) \tag{358}$$

b). Difference in potential we denote as V_0. Then $E(r) = \frac{V_0}{\phi_0 r}$. We also know that

$$\sigma(r) = 2\epsilon_0 E(r) \tag{359}$$

That is why the charge atone given plate is

$$q = d\int_b^{b+c} dr\sigma(r) = \frac{2\epsilon_0 V_0 d}{\phi_0} \log(1 + \frac{c}{b}) \tag{360}$$

The total charge is $Q = Nq$ where $N = \frac{2\pi}{2\phi_0}$ -number connected plates.

Now

$$Q = CV_0 = \frac{2\pi\epsilon_0 d}{\phi_0^2} \log(1 + \frac{c}{b}) V_0 \tag{361}$$

Capacity is

$$C = \frac{2\pi\epsilon_0 d}{\phi_0^2} \log(1 + \frac{c}{b}) = \frac{50\epsilon_0 d}{\pi} \log(1 + \frac{c}{b}) \tag{362}$$

if there are 10 plates of any kind.

2.23 e/e01m2 T

a).

$$m\ddot{x} = eE \tag{363}$$

Thus

$$x = \frac{eE}{mw^2} \tag{364}$$

$$P = nex \tag{365}$$

and

$$D = E + 4\pi P = \epsilon E \tag{366}$$

$$\epsilon = 1 - \frac{4\pi n e^2}{m w^2 \epsilon_0} = \frac{k^2}{w^2} \tag{367}$$

Thus

$$w_p^2 = \frac{4\pi n e^2}{m\epsilon_0} \tag{368}$$

b). It is smaller in approximately 2000 times due to $\frac{m_p}{m_e} \sim 2000$. c).

$$v_{ph} = \frac{w}{k} = \frac{c}{\sqrt{1 - \frac{w_p^2}{w^2}}} \tag{369}$$

$$v_{gr} = \frac{dw}{dk} = c\sqrt{1 - \frac{w_p^2}{w^2}} \tag{370}$$

and $v_{gr} v_{ph} = c^2$, $v_{gr} < c < v_{ph}$

d). Ok.

e).The equation to determine w_p (and n through it) is

$$\Delta T = \frac{l w_p^2 \Delta w}{c w^3} \tag{371}$$

eventually

$$n = \frac{m\epsilon_0 \Delta T c w^3}{4\pi e^2 l \Delta w} \tag{372}$$

2.24 e/e01m3 T

a). Electric field is directed along the radius. It is zero inside both cylinders. It is equal to

$$E(r) = \frac{b\sigma_b}{r\epsilon_0} \tag{373}$$

inside the larger cylinder and to

$$E(r) = \frac{b\sigma_b - a\sigma_a}{r\epsilon_0} \tag{374}$$

outside both of them.

b).$a\sigma_a = b\sigma_b$-total charge should be equal 0. c). This system is equal to solenoid with total current $I = \sigma_a a l w$ and field

$$H = \mu w_b b\sigma_b \tag{375}$$

inside. d). With the same frequency w. e). The flux inside the cylinders is

$$\Phi = \pi(b^2 - a^2)H \tag{376}$$

Then

$$\mathcal{E} = \dot{\Phi} = \pi(b^2 - a^2)\mu b\sigma_b \dot{w} \tag{377}$$

From the another side

$$\mathcal{E} = 2\pi b E \tag{378}$$

And additional torque is

$$M = 2\pi b l \sigma_b E b = \mu\pi(b^2 - a^2) l b^2 \sigma_b^2 \dot{w} \tag{379}$$

Now we see that this additional torque just shifts the moment of inertia of our system. f). At first

$$x \times \epsilon_0 E \times H = \mu a \sigma_a b \sigma_b w \tag{380}$$

and directed along axis of cylinders. Then

$$L = \int_a^b 2\pi r dr H = \mu a \sigma_a b \sigma_b w = \int dt M \tag{381}$$

This momentum corresponds to the integral of the torque.

2.25 e/e02j1 T

a).

$$v = \frac{c}{\sqrt{\epsilon\mu}} \tag{382}$$

$$Z = \frac{V}{I} = \frac{V}{\dot{Q}} \tag{383}$$

Since $Q = Q_0 e^{iwt - ikz}$ then

$$Z = \frac{-i}{vC} \tag{384}$$

where C is a usual capacity of the capacitor of given shape.

$$C = \frac{2\pi\epsilon}{\log(b/a)} \tag{385}$$

$$\frac{E}{B} = \frac{1}{\sqrt{\epsilon\mu}} \tag{386}$$

b). Kirgoff rule

$$I_0 + I_R = I_T \tag{387}$$

and

$$(I_0 - I_R) Z_1 = I_T Z_2 \tag{388}$$

Hence

$$I_R = I_0 \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{389}$$

and

$$I_T = I_0 \frac{2Z_1}{Z_1 + Z_2} \tag{390}$$

The same coefficients are applied for the electric field. c). Once again:

$$I_1 + I_2 = I_0 \tag{391}$$

and $I_1 = I_2$. And also

$$I_0(Z - R) = I_1(Z + R) \tag{392}$$

Thus

$$R = \frac{Z}{3} \tag{393}$$

2.26 e/e02j2 T

$$\mathcal{E} = \frac{S\dot{B}}{c} = 2\pi RE \tag{394}$$

$$M = RE\lambda = \frac{R^2\dot{B}\lambda}{2c} = I\dot{w} \tag{395}$$

Finally

$$w = \frac{R\Delta BT\lambda}{2Ic} \tag{396}$$

At the end of the story

$$B = \frac{2\lambda w}{c} \tag{397}$$

Now we have linear equation for w. Solving it we get a result

$$w = \frac{R\Delta BT\lambda}{2c[I + \frac{R^2\lambda^2}{c^2}]} \tag{398}$$

2.27 e/e02j3 T

Thing disk is equivalent to the plane circular wire with current $I = Mh$ in it. This at the z axis magnetic field (also directed along z) has value a).

$$B(z) = \frac{2\pi a^2 Mh}{(a^2+z^2)^{3/2}} \tag{399}$$

b). The force acting on the sphere is

$$\overrightarrow{F} = (m\cdot\partial)\overrightarrow{B} = (\mu-1)\frac{4\pi b^3}{3}B\partial_z B = (1-\mu)\frac{2\pi b^3}{3}\frac{24\pi^2 a^2 M^2 h^2 z}{(a^2+z^2)^4} \tag{400}$$

This force should be equal to mg.

2.28 e/e02m1 T

In the iron $B = \mu H$. In the gap $\tilde{B} = \tilde{H}$. From the low $divB = 0$ we have $B = \tilde{B}$ or $\tilde{H} = \mu H$.

Using circulation theorem:

$$NI = 2\pi RH + w\tilde{H} \tag{401}$$

or

$$H = \frac{NI}{2\pi R + \mu w} \tag{402}$$

Energy of the field is (here S is the area of cross-section)

$$E = \frac{S}{2}[2\pi RBH + w\tilde{H}\tilde{B}] = \frac{S\mu}{2}\frac{N^2I^2}{2\pi R + \mu w} \tag{403}$$

Obviously E decreases when w increases. Thus b). try to widen

a). $\tilde{B} = \frac{\mu NI}{2\pi R+\mu w}$

w could be taken $w = 0$ in the last formula

2.29 e/e02m2 T

At first it will useful to solve this problem for external field

$$E = E_0 \tag{404}$$

and for external field

$$E = E_0 \frac{\overrightarrow{x}}{a} \tag{405}$$

The eventual result could be obtained just using superposition rule.

The following useful formulas for $2D$ will be used:

$$\partial_\alpha \log r = \frac{n_\alpha}{r} \tag{406}$$

$$\partial_\alpha \partial_\beta \log r = \frac{\delta_{\alpha\beta} - n_\alpha n_\beta}{r^2} \tag{407}$$

and

$$\partial_\alpha \partial_\beta \partial_\gamma \log r = \frac{2}{r^3}[2n_\alpha n_\beta n_\gamma - \delta_{\alpha\beta} n_\gamma - \delta_{\alpha\gamma} n_\beta - \delta_{\gamma\beta} n_\alpha] \tag{408}$$

In both cases we assume that electric field inside the cylinder is directed along $\hat{x}$ (direction of external field). We denote it as $E(x)$. Then field on the boundary (outside the cylinder) is

$$\hat{x}[\cos^2\phi(\epsilon - 1)E + E] + \hat{y}[\cos\phi \sin\phi E(\epsilon - 1)] \tag{409}$$

In the case of homogeneous field E_0: the field inside the cylinder should be homogeneous (to satisfy $divD = 0$). This field should be equal external field plus field of polarized cylinder. The possible filed of polarized cylinder are given above. In this case we need $\partial_\alpha \partial_\beta \log r$-this substitution could solve the equation.

$$\hat{x}[\cos^2\phi(\epsilon - 1)E + E] + \hat{y}[\cos\phi \sin\phi E(\epsilon - 1)] = \hat{x}E_0 + \frac{A}{R^2}[\hat{x} - 2\hat{x}\cos^2\phi - 2\hat{y}\sin\phi\cos\phi] \tag{410}$$

Really we have two equation here (for $\hat{x}$ and for $\hat{y}$) and to unknown variables (A and E). The solution is

$$E = \frac{2E_0}{1+\epsilon} \tag{411}$$

$$A = -\frac{1}{2}E(\epsilon - 1)R^2 = -\frac{E_0(\epsilon - 1)}{1+\epsilon} \tag{412}$$

Now we are ready to present the results:

field outside the cylinder

$$\hat{x}E_0 - \frac{E_0(\epsilon - 1)R^2}{(\epsilon + 1)r^2}[\hat{x} - 2\hat{x}\cos^2\phi - 2\hat{y}\sin\phi\cos\phi] \tag{413}$$

field inside the cylinder

$$E = \frac{2E_0}{1+\epsilon} \tag{414}$$

D inside the cylinder:

$$D = \epsilon E = \frac{2E_0\epsilon}{1+\epsilon} \tag{415}$$

and P inside the cylinder

$$P = (\epsilon - 1)E = 2E_0\frac{\epsilon - 1}{\epsilon + 1} \tag{416}$$

Now we will consider the case of the field $E_0\frac{x}{a}$. Then there are free charge $\rho = \frac{E_0}{a} = divD$. And thus $D = E_0\frac{x}{a}$, $D = E_0\frac{x}{a}/\epsilon$.

On the boundary

$$\hat{x}[\cos^3\phi(\epsilon-1)\frac{E_0R}{\epsilon a}+\frac{E_0R\cos\phi}{\epsilon a}]+\hat{y}[\cos^2\phi\sin\phi\frac{E_0R(\epsilon-1)}{\epsilon a}] \tag{417}$$

This should be equal to

$$\hat{x}\frac{E_0R\cos\phi}{a} \tag{418}$$

plus field of polarized cylinder. We will use $\partial_\alpha\partial_\beta\partial_\gamma\log r\hat{x}^\beta\hat{x}^\gamma$ and also $\partial_\alpha\log r$ and $\hat{x}^\alpha\hat{x}^\beta\partial_\beta\log r$

The eventual result for electric field outside the cylinder is

$$\hat{x}\frac{E_0R\cos\phi}{a}-\frac{E_0R^2(1-\epsilon)}{2\epsilon ar}[\hat{x}\cos\phi+\hat{y}\sin\phi]-\frac{E_0R^4(1-\epsilon)}{2\epsilon ar^3}[2\cos^2\phi\{\hat{x}\cos\phi+\hat{y}\sin\phi\}-2\hat{x}\cos\phi-(\hat{x}\cos\phi+\hat{y}\sin\phi)] \tag{419}$$

Inside the cylinder electric field is

$$\hat{x}\frac{E_0x}{a\epsilon} \tag{420}$$

D is

$$\hat{x}\frac{E_0x}{a} \tag{421}$$

and P is

$$\hat{x}\frac{E_0x(\epsilon-1)}{a\epsilon} \tag{422}$$

2.30 e/e02m3 T

The density of protons is $\rho_0(r)=\frac{I}{v\pi r_0^2}$.

Let $\rho(x)$ be the density of electrons. Then electric field induced by electrons

$$E(r)=\frac{1}{r}\int_0^r d\xi\rho(\xi)2\pi\xi \tag{423}$$

The magnetic field

$$E(r)=\frac{v}{r}\int_0^r d\xi\rho(\xi)2\pi\xi \tag{424}$$

The electric field induced by positrons

$$E_(r)=\frac{1}{r}\int_0^r d\xi\rho_0 2\pi\xi \tag{425}$$

In the point where $\rho\neq 0$ al forces acting on the electron should compensate each other:

$$Ep=E-vB \tag{426}$$

or

$$\int_0^r d\xi\rho_0 2\pi\xi=(1-v^2)\int_0^r d\xi\rho(\xi)2\pi\xi \tag{427}$$

Result is

$$\rho(r)=\frac{1}{1-v^2}\rho_0 \tag{428}$$

From the other side we know that

$$\int_0^{r_0} d\xi\rho_0(\xi)2\pi\xi=\int_0^{r_0} d\xi\rho(\xi)2\pi\xi \tag{429}$$

Thus

$$\rho(r)=\frac{1}{1-v^2}\rho_0=\frac{I}{v\pi r_0^2(1-v^2)} \tag{430}$$

only for $r\in[0,r_0 sqrt 1-v^2]$ and 0 otherwise.

Voltage difference is

$$\int_0^{r_0} d\xi\rho_0\pi\xi-\int_0^{r\beta} d\xi\rho_0\beta^{-2}\pi\xi=0 \tag{431}$$

2.31 e/e03m1 T

a). Using that

$$A(r,t) = \mu I_0 \int_{-y}^{y} \frac{dx}{\sqrt{x^2+r^2}} \theta(ct-r) \tag{432}$$

where $y^2 + r^2 = c^2t^2$ we can determined the potential

$$A(r,t) = \frac{\mu I_0}{2} \log[\frac{ct+\sqrt{c^2t^2-r^2}}{ct-\sqrt{c^2t^2-r^2}}]\theta(ct-r) \tag{433}$$

Then

$$|B(r,t)| = |\partial_r A| = \frac{\mu I_0 ct}{r\sqrt{c^2t^2-r^2}}\theta(ct-r) \tag{434}$$

The force (repealing) per unit length is

$$F = \frac{\mu I_0^2 ct}{r\sqrt{c^2t^2-r^2}}\theta(ct-r) \tag{435}$$

At the moment $t = r/c$ (the first moment e.m. wave reach second wire the force is infinite) because we turn on the current unsmoothly. b). Similarly to the previous case ($\tau = \frac{I_0}{b}$, $z^2 + r^2 = c^2(t-\tau)^2$)

$$A(r,t) = \mu I_0 \int_{-z}^{z} \frac{dx}{\sqrt{x^2+r^2}}\theta(c(t-\tau)-r) + 2\mu \int_{z}^{y} \frac{dxb(t-\sqrt{x^2+r^2}/c)}{\sqrt{x^2+r^2}} \tag{436}$$

Then B

$$B(r,t) = \frac{\mu br\theta(t-r/c)}{c(ct+\sqrt{c^2t^2-r^2})} + \frac{\mu\theta(t-\tau-r/c)}{\sqrt{c^2t^2-r^2}}[\frac{brt}{c(t-\tau)+\sqrt{c^2(t-\tau)^2-r^2}} - \frac{br}{c} - \frac{b\tau c(t-\tau)}{r}] \tag{437}$$

And force is $F = I(t)B$. Since we turn on current smoothly the force is smooth as well.

2.32 e/e03m2 T

a). If $q = -\frac{a}{R}Q$ and is located on the line between center of the sphere and charge Q at the distance $r = \frac{a^2}{R}$ from center of sphere towards charge then V on sphere is zero.

b).By Newton's third low the force acting on the sphere is equal (abs. value) to the force acting on the charge Q. Conducting sphere with charge Q on it could be substituted by the point charge Q locate in the center of the sphere and conducting sphere without any charge. The last one could be substituted by the charge specified in a. Thus the forces are

attraction

$$\frac{kqQ}{(R-r)^2} = \frac{akQ^2}{R^2-a^2} \tag{438}$$

repealing

$$\frac{kQ^2}{R^2} \tag{439}$$

Result is the repealing force

$$\frac{kQ^2}{R^2}[1 - \frac{aR^3}{(R^2-a^2)^2}] \tag{440}$$

c). The electrical potential at the distance l from the center of the sphere and at the angle ϕ ($\phi = 0$ is a symmetry axis) is

$$\varphi(l,\phi) = kQ[\frac{1}{l} + \frac{1}{\sqrt{R^2+l^2-2Ra\cos\phi}} - \frac{1}{\sqrt{a^2+\frac{R^2}{a^2}l^2-2Ra\cos\phi}}] \tag{441}$$

Thus electrical field on the surface $l = a$ is

$$E = -\frac{\partial \varphi}{\partial l} = kQ[\frac{1}{a^2} - \frac{a(R^2 - a^2)}{(R^2 + a^2 - 2Ra\cos\phi)^{3/2}}] \quad (442)$$

We know that density of the charge at the surface of conductor is $\sigma = E\epsilon_0$ where $k = \frac{1}{4\pi\epsilon_0}$ Result

$$\sigma(\phi) = \frac{Q}{4\pi}[\frac{1}{a^2} - \frac{a(R^2 - a^2)}{(R^2 + a^2 - 2Ra\cos\phi)^{3/2}}] \quad (443)$$

Obviously negative charge will first appear at $\phi = 0$ when $R = 3a$.

d). At zero approximation (only one sphere with potential V_0 on it) the charge on the sphere is $Q_0 = 4\pi\epsilon_0 a V_0$. Next approximation: we have one sphere very far from other and could consider it as a point charge of Q_0 value. Then second sphere has charge $Q_1 = Q_0 - \frac{a}{R}Q_0 = Q_0\frac{R-a}{R}$.

e). At the next step we have to consider on sphere as the point charge Q_0 in it's center and once more charge $-\frac{a}{R}Q_0$ at the distance $\frac{a^2}{R}$. Since this distance is already quadratic in a we can neglect it and repeat our speculations from d. substituting Q_1 instead of Q_0. Result is

$$Q = Q_0[1 - \frac{a}{R} + \frac{a^2}{R^2}]$$

2.33 e/e03m3 T

The vector, pointing charged particle (in the x-y plane) is

$$\vec{r} = -\frac{e\vec{E}}{mw^2} \quad (444)$$

By definition

$$\vec{P} = N\vec{d} \quad (445)$$

where $\vec{d}$ is dipole moment of produced of one (negatively) charged particle: $\vec{d} = e\vec{r}$ Now

$$\epsilon(w) = \epsilon_0(1 - \frac{e^2N}{\epsilon_0 m w^2}) \quad (446)$$

and

$$n(w) = \sqrt{1 - \frac{e^2N}{\epsilon_0 m w^2}} \quad (447)$$

As well known speed of light in the medium is $c = 1/n = \frac{w}{k}$ or

$$\frac{k^2}{w^2} = 1 - \frac{e^2N}{\epsilon_0 m w^2} \quad (448)$$

Dispersion low is

$$w(k) = \sqrt{k^2 + \frac{e^2N}{\epsilon_0 m}} \quad (449)$$

Obviously the plasma frequency is

$$w_p = \sqrt{\frac{e^2N}{\epsilon_0 m}} \quad (450)$$

b). Equation of motion (for complex velocity vector)

$$\dot{v} + w_L v = \frac{eE}{m} \quad (451)$$

Solving this one gets

$$r = -\frac{eE(z,t)}{w(w+iw_L)} + r_0 e^{-iw_L t} \tag{452}$$

Hence

$$\epsilon = \epsilon_0 (1 - \frac{e^2 N}{\epsilon_0 m w(w+iw_L)} + \frac{eNr_0}{\epsilon_0}) \tag{453}$$

(As I understand phase of α is just the phase of r_0)

c). The minimal frequency should satisfy equation

$$(1 - \frac{e^2 N}{\epsilon_0 m w(w+iw_L)} + \frac{eNr_0}{\epsilon_0}) = 0 \tag{454}$$

3 Quantum Mechanics

3.1 q/q98j1 V

If the the hamiltonian H_0 is perturbed by V then it is possible to find the expansion of the eigenstates $|i\rangle$of $H = H_0 + V$ over eigenstates of H_0, which are denoted $\left|i^0\right\rangle$ and have energy E_i^0:

$$|i\rangle = \sum_j c_{ij} \left|j^0\right\rangle \tag{455}$$

$$(H_0 + V) \sum_j c_{ij} \left|j^0\right\rangle = E_i \sum_j c_{ij} \left|j^0\right\rangle \tag{456}$$

In the zero order $c_{ij}^0 = \delta_{ij}$. In the first order

$$c_{ij}^1 = \delta_{ij} - \frac{V_{ji}}{E_j^0 - E_i^0} \tag{457}$$

Thus the expectation value of the operator $\mathcal{O}$ in the state $|s\rangle$ is

$$\langle s| \mathcal{O} |s\rangle = \mathcal{O}_{ss} - \sum_j \frac{\mathcal{O}_{sj} V_{js} + \mathcal{O}_{js} V_{sj}}{E_j^0 - E_0^0} + O(V^2) \tag{458}$$

where the matrix elements are meant to be taking over the states $\left|i^0\right\rangle$. The ground state is doubly degenerate as well as the first excited state. However, in the sum the contribution will be only from the diagonal term. Since for the oscillator

$$x_{n,n-1} = \sqrt{\frac{n\hbar}{2m\omega}} \tag{459}$$

we get the answer:

$$\langle 0| z\sigma_z |0\rangle = -\frac{\lambda}{\hbar\omega} \frac{\hbar}{m\omega} \tag{460}$$

for any of two ground states.

3.2 q/q98j2 V

From

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + a\delta(x)\psi = E\psi \tag{461}$$

it follows that the jump of the derivative at each delta function is

$$\psi'|_-^+ = \alpha\psi \tag{462}$$

where $\alpha = \frac{2m\alpha}{\hbar^2}$. The solution to be found in form $Ae^{-k|x|}, B\sinh(kx+f), Ce^{-k|x|}$ for the regions to the left, between and to the right of the bands respectively. Gluing the logarithmic derivatives we get the system of transcendental equations

$$k - k\coth(-kL+f) = \alpha \tag{463}$$

$$-k - k\coth(kL+f) = \alpha \tag{464}$$

from which the solution for k and the ground state energy $E = -\frac{\hbar^2 k^2}{2m}$ can be found. In the limit $L \to 0$ we have

$$k\coth f = \alpha \tag{465}$$

Expanding $\coth x$ near this point we get

$$\sinh^2 f = kL \tag{466}$$

from which follows the answer

$$k = \frac{2mLa}{\hbar^2} \tag{467}$$

$$E = -\frac{\hbar^2 k^2}{2m} \tag{468}$$

3.3 q/q98j3 V

Since $S_e S_p = \frac{1}{2}((S_e+S_p)^2 - S_e^2 - S_p^2)$ the eigenstates coincide with the eigenstates of the total momentum $S = S_e + S_p$, which can take value 0 or 1 with degeneracies 1 and 3 respectively. The eigenvalues, respectively, are:

$$\frac{1}{2}\left(S(S+1) - \frac{3}{2}\right) = -\frac{3}{4}a\hbar^2, \frac{1}{4}a\hbar^2 \tag{469}$$

The interaction of the magnetic field with the spin $\frac{1}{2}$ particle is $\frac{g}{2}\frac{e\hbar}{2mc}\sigma^i B_i$, where is the gyromagnetic ration (dimensionless number) and $g = 2$ for nonrelativistic electron. Since $m_p \gg m_e$ the interaction of it with the magnetic field could be neglected in this problem. In the basis $|0,0\rangle, |1,0\rangle, |1,1\rangle, |1,-1\rangle$ in the notation $|J,M\rangle$ for the total momentum the hamiltonian takes the form

$$\frac{1}{\hbar}H = \begin{pmatrix} -\frac{3}{4}\alpha & \frac{1}{2}\mu & 0 & 0 \\ \frac{1}{2}\mu & \frac{1}{4}\alpha & 0 & 0 \\ 0 & 0 & \frac{1}{4}\alpha & 0 \\ 0 & 0 & 0 & \frac{1}{4}\alpha \end{pmatrix}, \qquad \alpha = a\hbar, \qquad \mu = \frac{eB}{2mc} \tag{470}$$

and the evolution operator $e^{-i\frac{Ht}{\hbar}}$ restricted to the upper left block, where is ground state, is

$$e^{\frac{1}{4}it\alpha}\left(\left[-i\sigma_1\frac{\mu}{\sqrt{\mu^2+\alpha^2}} + i\sigma_3\frac{\alpha}{\sqrt{\mu^2+\alpha^2}}\right]\sin\Omega t + \sigma_0\cos\Omega t\right), \qquad \Omega = \frac{1}{2}\sqrt{\mu^2+\alpha^2} \tag{471}$$

and the probability to remain in the ground state is

$$P = 1 - \frac{\mu^2}{\mu^2+\alpha^2}\sin^2\Omega t \tag{472}$$

3.4 q/q98m1 V

a) The ground state is $|g\rangle = |m_1\rangle |m_2\rangle$ with $m_1 = J_1, m_2 = -J_2$ to maximize $S_{1z} - S_{2z}$.

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2(S_{1z}S_{2z} + \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+})) \tag{473}$$

Only first three terms contribute into expectation value on the ground state, and thus

$$\langle g| S^2 |g\rangle = J_1(J_1+1) + J_2(J_2+1) - 2J_1J_2 \tag{474}$$

Using matrix elements of operators

$$\langle m+1| L_+ |m\rangle = \sqrt{(l-m)(l+m+1)}, \qquad \langle m-1| L_- |m\rangle = \sqrt{(l-m+1)(l+m)} \tag{475}$$

it is easy to decompose the tensor product of $SU(2)$ spin J_1, J_2 representation into the direct sum of the irreducible ones (the adding of moments). The notations below are $|J, M\rangle$ for the total momentum, and $|m_1\rangle |m_2\rangle$ for separate projections of the particle angular momentum on the quantization axis.

b) For $J_1 = 1, J_2 = \frac{1}{2}$ starting from the highest weight and acting consequently by L_- we get

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |1\rangle \left|\frac{1}{2}\right\rangle \tag{476}$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|0\rangle \left|\frac{1}{2}\right\rangle + |1\rangle \left|-\frac{1}{2}\right\rangle\right) \tag{477}$$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle \left|\frac{1}{2}\right\rangle - \sqrt{2}\,|1\rangle \left|-\frac{1}{2}\right\rangle\right) \tag{478}$$

Reverting the expansion we get

$$|1\rangle \left|-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\left|\frac{3}{2}, \frac{1}{2}\right\rangle - \sqrt{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right) \tag{479}$$

The outcomes are $J(J+1) = \{\frac{15}{4}, \frac{3}{4}\}$ for $J = \frac{3}{2}, \frac{1}{2}$ correspondingly with the probabilities $\frac{1}{3}, \frac{2}{3}$.

3.5 q/q98m2 V

Use the Born approximation to consider scattering on the potential $V(r) = V_0 e^{-(r/a)^2}$.

$$d\sigma = \frac{2\pi}{v\hbar}\int \frac{d^3k}{(2\pi)^3}\delta(E_i - E_f)\left|\int d^3x V(x) e^{iqx}\right|^2 \tag{480}$$

using

$$d^3k = k^2 d\Omega \frac{mdE}{k\hbar^2}, \tag{481}$$

and

$$\int d^3x V(r) e^{iqx} = \frac{2\pi}{q}\int_0^\infty r dr V(r) \sin qr \tag{482}$$

and

$$\int_0^\infty r dr e^{-\frac{r^2}{a^2}} \sin qr = \frac{\sqrt{\pi}}{4} a^3 q e^{-\frac{q^2a^2}{4}} \tag{483}$$

we get

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar v}\frac{1}{(2\pi)^3}\frac{k^2 m}{\hbar^2}\frac{1}{k}\left(V_0 \frac{2\pi\sqrt{\pi}a^3}{4} e^{-\frac{q^2a^2}{4}}\right)^2 \tag{484}$$

b) Integrating over angles is straightforward with $q = 2k \sin\frac{\theta}{2}$ and $qdq = k^2 \sin\theta \partial theta$ and the result is

$$\sigma = \frac{2\pi}{\hbar v} \frac{1}{(2\pi)^3} \frac{k^2 m}{\hbar^2} \frac{1}{k} V_0^2 \frac{\pi^3 a^6}{4} \frac{2\pi}{k^2 a^2} \left(1 - e^{-\frac{q_{max}^2 a^2}{2}}\right) \tag{485}$$

with $q_{max} = 2k$

$$\sigma = \frac{\pi^2 V_0^2 a^4}{8 v^2 \hbar^2} (1 - e^{-2k^2 a^2}) \tag{486}$$

c) The approximation is valid when $V_0 \ll \frac{m\hbar^2}{a^2}$ or when $V_0 \ll \frac{m\hbar^2}{a^2} qa$. Also the equation for $d\sigma$ is not justified when $|qa| \gg 1$ (see Landau-Lifshitz).

3.6 q/q98m3 V

a) From the perturbation theory we know that the second order perturbation of the ground state is always negative:

$$\delta E = -\sum \frac{|V_{0n}|^2}{E_n - E_0} \tag{487}$$

which means that the second derivative over the parameter is negative (the first derivative is given by the first order perturbation theory). If one wants to use hint it is also straightforward. Take point $\tilde{\lambda}$ and the ground state $|\lambda\rangle$, with ground energy $E(\tilde{\lambda})$. Then, by since the ground state should minimize the functional $\langle\psi| H |\psi\rangle$ we get

$$E(\lambda) \le \left\langle \tilde{\lambda} \middle| H_1 + \lambda H_2 \middle| \tilde{\lambda} \right\rangle = \left\langle \tilde{\lambda} \middle| H_1 + \tilde{\lambda} H_2 \middle| \tilde{\lambda} \right\rangle + (\lambda - \tilde{\lambda}) \left\langle \tilde{\lambda} \middle| H_2 \middle| \tilde{\lambda} \right\rangle = E(\tilde{\lambda}) + (\lambda - \tilde{\lambda}) \left\langle \tilde{\lambda} \middle| H_2 \middle| \tilde{\lambda} \right\rangle \tag{488}$$

Therefore

$$E(\lambda) - E(\tilde{\lambda}) \le (\lambda - \tilde{\lambda}) \left\langle \tilde{\lambda} \middle| H_2 \middle| \tilde{\lambda} \right\rangle = (\lambda - \tilde{\lambda}) E'(\tilde{\lambda}) \tag{489}$$

and this means concavity.

b)

For $a = 0$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & b & 1 \end{pmatrix} \tag{490}$$

The eigenvalues are $\{1, 1 \pm b\}$. Take for example $b > 0$ the opposite case is analogous. The ground energy is $1 - b$ and the ground state is $(0, 1, -1)$. The first order perturbation is given by the matrix

$$\begin{pmatrix} 0 & 1 & b \\ 1 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \tag{491}$$

which convoluted with ground state gives zero. Thus, the first derivative is zero at $a = 0$. The second will be always negative. Therefore the energy will decrease as a increases.

3.7 q/q99j1 V

In the non relativistic approximation there is no spin-spin or spin-orbital interaction. Therefore the wave function can be written as a direct product of the spin wave function by the coordinate wave function. Two spin 1/2 particles can combine either in the total momentum 0 state with the symmetrical spin function or in the total momentum 1 state with the antisymmetrical spin function. Since the total wave function should antisymmetrical because of grassmanian nature of the fermions, we obtain that the coordinate wave function should be symmetrical for $S = 0$ state and antisymmetrical for $S = 1$ state. The angular momentum in the ground state of the hydrogen molecule is zero. The corresponding terms of the hydrogen molecule are denoted by ${}^1\Sigma_g^+$ and ${}^3\Sigma_u^+$ respectively.

$$\left|{}^1\Sigma_g^+\right\rangle = \chi_{\mu\nu}^{anti} \phi^{sym}(r_1, r_2) \tag{492}$$

$$\left|{}^3\Sigma_u^+\right\rangle = \chi_{\mu\nu}^{sym}\phi^{anti}(r_1, r_2) \tag{493}$$

In each case, in the perturbation theory as a trial coordinate wave functions we can use the symmetrized/antisymmetrized products of the wave functions of the separate atoms. Denote protons by the letters A, B, and electrons by the numbers $1, 2$.

$$H = K_1 + K_2 + U_A(r_1) + U_B(r_1) + U_A(r_2) + U_B(r_2) + U_{12}(r_1, r_2) + U_{pp} \tag{494}$$

where $K_i = -\frac{\hbar^2}{2m_e}\Delta_i$ is kinetic energy, $U_{A,B}(r_i) = -\frac{e^2}{|r_i - r_{A,B}|}$ is the interaction between the electrons and protons, $U_{12} = \frac{e^2}{|r_1 - r_2|}$ is the interaction between the electrons, and $U_{AB} = \frac{e^2}{|r_A - r_B|}$ is the interaction between the protons.

The trial wave functions are:

$$|\pm\rangle = \frac{1}{\sqrt{2}}(\phi_A(r_1)\phi_B(r_2) \pm \phi_A(r_2)\phi_B(r_1)) \equiv \frac{1}{\sqrt{2}}(|AB\rangle \pm |BA\rangle) \tag{495}$$

In the first order perturbation theory the difference between energies of the $|\pm\rangle$ states is given by the difference of the matrix elements $\langle\pm|\, H \,|\pm\rangle$.

b) Plugging the expressions for $|\pm\rangle$ to the Hamiltonian we get the difference

$$\delta E = \langle +|\, H \,|+\rangle - \langle -|\, H \,|-\rangle = \langle AB|\, 4E + 2U_{pp} \,|BA\rangle + \langle BA|\, U_{BA} + U_{12} \,|AB\rangle + \langle AB|\, U_{AB} + U_{12} \,|BA\rangle\,, \tag{496}$$

Since it is composed from the matrix elements of the wave functions for the separate atoms the overlapping of them is proportional to $e^{-R_{AB}}$.

c) and the mean energy

$$E_m = \frac{1}{2}(\langle +|\, H \,|+\rangle + \langle -|\, H \,|-\rangle) = 2E + \langle AB|\, U_{BA} + U_{12} + U_{pp} \,|AB\rangle + \langle BA|\, U_{AB} + U_{12} + U_{pp} \,|AB\rangle \tag{497}$$

where $U_{AB} = U_A(r_1) + U_B(r_2)$ and $U_{BA} = U_B(r_2) + U_A(r_1)$. The cross terms represent interaction between between atoms. In the dipole approximation the interaction energy is

$$\frac{(d_1 R_{AB})(d_2 R_{AB}) - d_1 d_2 R_{AB}^2}{R_{AB}^5} \tag{498}$$

and therefore is proportional to $d_1 d_2 / R^3$. But the average dipole moment vanishes. Therefore, only the second order perturbation gives nonvanishing result, that is $U_{eff}(R_{AB}) \sim R_{AB}^{-6}$.

3.8 q/q99j2 V

It is convenient to represent the Hamiltonian in the following form

$$H = \frac{1}{2}((S_1 + S_2 + S_3 + S_4)^2 - (S_1 + S_3)^2 - (S_2 + S_4)^2) = \frac{1}{2}(J(J+1) - J_{13}(J_{13}+1) - J_{24}(J_{24}+1)) \tag{499}$$

Then we can classify all states by consequent adding of momentums. First add in the pairs $J_{13} = S_1 + S_3$ and $J_{24} = S_2 + S_4$ and then add the pairs.

The result $J_{13} \times J_{24} \rightarrow^{2J+1} |J\rangle$, where J is the total momentum and the degeneracy is $2J+1$.

$0 \times 0 =^1 |0\rangle \qquad H = 0$

$1 \times 0 =^3 |1\rangle \qquad H = 0$

$0 \times 1 =^3 |1\rangle \qquad H = 0$

$1 \times 1 =^1 |0\rangle, ^3|1\rangle, ^5|2\rangle \qquad H = -2, -1, 1.$

For any spin the ground state is obtained in the variant with spins added to the maximal one. So, it has angular momentum $4J$, the degeneracy $8J+1$, and the energy

$$\frac{1}{2}(4J(4J+1) - 4J(2J+1)) \tag{500}$$

3.9 q/q99j3v V

In the limit $V_0 \gg \frac{\hbar^2 k^2}{2m}$ the probability of transition

$$w = \frac{2\pi}{\hbar}\frac{m}{|\hbar p|}\frac{1}{2\pi\hbar}\frac{2}{a}\left[\left|\int_0^a e^{i\tilde{p}x}\frac{1}{2}\varepsilon_0 x\sin kx\right|^2 + \left|\int_0^a e^{-i\tilde{p}x}\frac{1}{2}\varepsilon_0 x\sin kx\right|^2\right], \tag{501}$$

where

$$ka = \pi, \qquad -V_0 + \frac{\hbar^2 k^2}{2m} + \hbar\omega = \frac{\hbar^2 p^2}{2m}, \qquad \frac{\hbar^2 k^2}{2m} + \hbar\omega = \frac{\hbar^2 \tilde{p}^2}{2m} \tag{502}$$

and

$$\int_0^a e^{i\tilde{p}x}\sin kx = \frac{a\tilde{p}}{k^2 - \tilde{p}^2} \pm \frac{1}{(\tilde{p}\pm k)^2}(e^{i(\pm\tilde{p}+k)} - 1) = -\frac{a}{\tilde{p}} + O\left(\frac{1}{\tilde{p}^2}\right) \tag{503}$$

is computed to be

$$w = \frac{2\pi}{\hbar}\frac{m}{|\hbar p|}\frac{1}{2\pi\hbar}\frac{2}{a}\frac{1}{2}\varepsilon_0^2\frac{a^2}{\tilde{p}^2} = \frac{\varepsilon_0^2 a}{2\hbar\omega\sqrt{2m(\hbar\omega - V_0)}} \tag{504}$$

3.10 q/q99m1 V

The hamiltonian is

$$H = \alpha(\overrightarrow{S_1}\overrightarrow{S_2} - 3S_{1z}S_{2z}) \tag{505}$$

where S are half-pauli matrices and

$$\alpha = \frac{\hbar^2 e^2}{m^2 c^2 L^3} \tag{506}$$

The eigenstates are

$$|{++}\rangle, |{--}\rangle, \frac{1}{\sqrt{2}}(|{-+}\rangle + |{+-}\rangle), \frac{1}{\sqrt{2}}(|{+-}\rangle - |{-+}\rangle) \tag{507}$$

with the eigen values $\alpha(-1/2, -1/2, 1, 0)$ respectively.

a) The initial state $|{++}\rangle$ is eigenstate. Therefore it only acquires the phase $e^{-\frac{i\alpha t}{\hbar}}|{++}\rangle$ and the result of measuring $S_{1z} + S_{2z}$ is always 1.

b) The initial state is

$$\frac{1}{2}(|+\rangle_1 + |-\rangle_1)(|+\rangle_2 + |-\rangle_2) = \frac{1}{2}(|{++}\rangle + |{--}\rangle + |{+-}\rangle + |{-+}\rangle), \tag{508}$$

which evolves to

$$\frac{1}{2}(e^{\frac{i\alpha t}{2\hbar}}|{++}\rangle + e^{\frac{i\alpha t}{2\hbar}}|{--}\rangle + e^{-\frac{i\alpha t}{\hbar}}|{+-}\rangle + |{-+}\rangle), \tag{509}$$

Projecting onto eigenvectors of $S_{1x} + S_{2x}$ we get that the outcome with the result 1 is possible with the probability $\cos^2(\frac{3}{4}\frac{\alpha t}{\hbar})$ and the result with the outcome -1 is possible with the probability $\sin^2(\frac{3}{4}\frac{\alpha t}{\hbar})$

c) The classical dipoles rotates smoothly with preserving the same direction for the a)-case and rotating with its projection to the X-axis equal to the expectation value $\cos(\frac{3}{2}\frac{\alpha t}{\hbar})$ in the b) case.

3.11 q/q99m2 V

The electronic configuration of different electrons is specified approximately by showing which orbit each electron occupies with notation like $1s2s$ or $1s2p$. The total configuration is given by specifying the total S spin of the orbital and the total L momentum of the orbital. They conserve separately in the "LS-approximation" but are perturbed by relativistic LS-interaction. The exactly conserved quantity is the total momentum $J = L + S$. Given L and S the resulting J could run from $|L - S|$ to $L + S$, and for each J the corresponding state is degenerated $2J + 1$ times over the directions of J. The configuration is displayed

$$^{2S+1}L_J \tag{510}$$

where instead of $L = 0, 1, 2, 3$ the letters $S, P, D, F, ..$ are used.

Two electrons can combine into total spin S equal to 1 or to 0. The L is defined by the second electron, which in s-state gives 0 and in p-state gives 1.

The possible states in the problem are:

$^1S_0, ^3S_1$ for the second electron in the $2s$ shell. $^1P_1, ^3P_0, ^3P_1, ^3P_2$ for the second electron in the $2p$-shell. The degeneracies are $2J + 1$, which gives respectively $1, 3, 3, 1, 3, 5$.

The lowest energy should be S-state ($L = 0$). The total spin probably should be 1 than the spin wave function is symmetrical and the space-time wave function is antisymmetrical which decrease the energy due to repulsions of electrons. The Hund rule states that the minimal energy of the states with the same electronic configurations in terms of filing n, l-shells has the term with:

First $S \to max$, then $L \to max$

Which agrees with the proposal for the minimum energy of $^3S_1$-state.

May be the maximal energy will be of the term $^1P_1$ for the same reason.

The strongest decay process should be due to the electric dipole radiation $E1$ from $2p$ shell to $1s$ shell: $^1P_1 \to ^1S_0$.

The $^3P_{0,1,2}$ state could decay into $^3S_1$ due to the $E1$ process. Just for the reference the probability is

$$w = \frac{4\omega^3}{3\hbar c^3}|d_{fi}|^2. \tag{511}$$

It seems, that the decays from $^1S_0$ and $^3S_1$ states are impossible due to $E1$-process (which is space-parity negative). The parity of E-photon is $(-1)^j$, the parity of M-photon is $(-1)^{j+1}$. But from $^3S_1$ it seems possible to do with one electron emitted in $M1$-wave (which is space-parity positive wave). From $^1S_0$ state the 1-photon decay seems to be impossible. That should be the longest level. (And it is a little bit upper $^3S_1$?).

3.12 q/q99m3 V

The Alice photon 1 is $\alpha|+\rangle_1 + \beta|-\rangle_1$. Entangled with the photon $|\Phi\rangle_{2,3} = |+,-\rangle_{2,3} - |-,+\rangle_{2,3}$ it gives rise to the 3-photon state $\alpha(|++-\rangle_{123} - |+-+\rangle_{123}) + \beta(|-+-\rangle - |--+\rangle)$. Given orthogonal basis of states for the 1st and the 2nd photon, we can expand the total 3-photon wave function over this basis, where the coefficients of the expansion are the wave functions of the 3-rd photon. After measurement the state falls into that basis vector which was measured. Since the basis of states for the 1st and the 2nd photon is the orthogonal the coefficients up to a scale can be found by taking the scalar product. For each case $j = a, b, c, d$ the relation

$$|\phi\rangle_3^j = \langle\Phi|_{1,2}^j(\alpha(|++-\rangle_{123} - |+-+\rangle_{123}) + \beta(|-+-\rangle - |--+\rangle)) \tag{512}$$

after explicit substitution gives

$$|\phi\rangle_3^a = -\alpha|+\rangle_3 - \beta|-\rangle_3 \tag{513}$$

$$|\phi\rangle_3^b = -\alpha|+\rangle_3 + \beta|-\rangle_3 \tag{514}$$

$$|\phi\rangle_3^c = +\beta|+\rangle_3 + \alpha|-\rangle_3 \tag{515}$$

$$|\phi\rangle_3^d = -\beta\,|+\rangle_3 + \alpha\,|-\rangle_3 \tag{516}$$

Therefore, to obtain the same state from the 3rd photon that Alice initially had for the 1st photon, Bob has to rotate the incoming photon by the matrices $-\sigma_0, -\sigma_3, \sigma_1, -i\sigma_2$ correspondingly to the a,b,c,d cases.

3.13 q/q00j1 V

$$d\sigma = \frac{1}{v}\frac{2\pi}{\hbar}|\left\langle e^{-ik'x}V(x)e^{ikx}\right\rangle|^2\delta(E_f - E_i)\frac{d^3k}{(2\pi)^3} \tag{517}$$

$$d^3k = d\Omega k^2\frac{mdE}{k\hbar^2} \tag{518}$$

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\hbar^4 q^2}|\int_0^\infty drV(r)r\sin qr|^2, \qquad \int_0^\infty \frac{r\sin qr}{r^2+a^2}dr = \frac{1}{2}\pi e^{-qa} \tag{519}$$

$$V(r) = \frac{b}{r^2+a^2}, \qquad b = \frac{\hbar^2}{\pi m}\sqrt{A}, \qquad a = \lambda \tag{520}$$

$$\frac{d\sigma}{d\Omega} = \frac{A}{q^2}\exp(-2\lambda q) \tag{521}$$

3.14 q/q00j2 V

$$\delta E = -\sum\frac{|V_{0n}|^2}{E_n - E_0}, \qquad \langle n-1|x|n\rangle = \sqrt{\frac{n\hbar}{2m\omega}} \tag{522}$$

There are three non-vanishing matrix elements $\langle 000000V100100\rangle, \langle 000000V010010\rangle, \langle 000000V001001\rangle$:

$$\delta E = -\frac{1}{2\hbar\omega}\epsilon^2\frac{\hbar^2}{(2m\omega)^2}(1+1+4) \tag{523}$$

3.15 q/q00j3 V

In the plane xy, while B is along z and $B_z = B$:

$$A = -\frac{1}{2}[rB], \quad B = \text{rot}\,A, \quad A_\phi = \frac{1}{2}rB, \quad A_r = 0 \tag{524}$$

$$H = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 \tag{525}$$

where p is the momentum $-i\hbar\partial$ of a free particle. Then eigenvalues of p on the circle are $\frac{\hbar n}{R}$. The total answer

$$E = \frac{\hbar^2}{2mR^2}(n - \frac{R^2eB}{2c\hbar})^2 \pm \frac{e\hbar B}{2mc} \tag{526}$$

3.16 q/q00m1 T

q00m1 Problem 1.

The energy of the state $|n,s>$ is $w(n+s-1/2)$ $(\hbar = 1)$.

Thus the levels $|A>=|n,s>=|-1/2>$ and $|B>=|n+1,s>=|1/2>$ are degenerate. At the first order of perturbation theory the splitting is the difference between eigenvalues of the perturbation hamiltonian $H_1 = \alpha x\hat{S}_x$

$$\begin{pmatrix} <A|H_1|A> & <A|H_1|B> \\ <B|H_1|A> & <B|H_1|B> \end{pmatrix} = \begin{pmatrix} 0 & \alpha\sqrt{\frac{n}{8mw}} \\ \alpha\sqrt{\frac{n}{8mw}} & 0 \end{pmatrix} \tag{527}$$

And the splitting is just $\alpha\sqrt{\frac{n}{2mw}}$

3.17 q/q00m2 T

q00m2
Let $\overrightarrow{B}$ be along z axis. Then consider for states

$$|\frac{1}{2}>\otimes|\frac{1}{2}>$$

$$|-\frac{1}{2}>\otimes|-\frac{1}{2}>$$

$$\frac{1}{\sqrt{2}}(|\frac{1}{2}>\otimes|-\frac{1}{2}>-|-\frac{1}{2}>\otimes|\frac{1}{2}>)$$

$$\frac{1}{\sqrt{2}}(|\frac{1}{2}>\otimes|+\frac{1}{2}>+|-\frac{1}{2}>\otimes|\frac{1}{2}>)$$

It is also useful to express $2\overrightarrow{s}_e\overrightarrow{s}_p=(s_e+s_p)^2-3/2$ First three states are already eigenvectors with eigenvalues $\alpha\pm|B|(\beta+\gamma)$ In the basis of the last two vectors hamiltonian has the form

$$\begin{pmatrix} -3\alpha & (\beta-\gamma)|B| \\ (\beta-\gamma)|B| & \alpha \end{pmatrix} \tag{528}$$

And the eigenvalues of this matrix are $-(\alpha\pm\sqrt{(2\alpha)^2+((\beta-\gamma)|B|)^2})$

3.18 q/q00m3 T

q00m3 At let us remind the explicit form of eigenvectors of angular momentum 1

$$|1>=\frac{3}{8\pi}e^{i\varphi}Sin\theta$$

$$|0>=\frac{3}{4\pi}Cos\theta$$

$$|-1>=\frac{3}{8\pi}e^{-i\varphi}Sin\theta$$

Splitting of representation **2** into two **1** has the form

$$|2,2>=|1>\otimes|1>$$

$$|2,1>=\frac{1}{\sqrt{2}}(|1>\otimes|0>+|0>\otimes|1>)$$

$$|2,0>=\frac{1}{\sqrt{6}}(|1>\otimes|-1>+|-1>\otimes|1>-2|0>\otimes|0>)$$

$$|2,-1>=\frac{1}{\sqrt{2}}(|-1>\otimes|0>+|0>\otimes|-1>)$$

$$|2,-2>=|-1>\otimes|-1>$$

This result is the simple consequences of the symmetry constraint and of the constraint that $|2,0>$ should be orthogonal to

$$|0,0>=\frac{1}{\sqrt{3}}(|1>\otimes|-1>+|-1>\otimes|1>+|0>\otimes|0>)$$

If we are interesting only in $\theta_1=\theta_2=\pi/2$ then

$$|2,2>=\frac{3}{8\pi}e^{i(\varphi_1+\varphi_2)}$$

$$|2,1>=0$$

$$|2,0>=\frac{3}{8\pi}\frac{1}{\sqrt{6}}(e^{i(\varphi_1-\varphi_2)}+e^{-i(\varphi_1-\varphi_2)})$$

$$|2,-1>=0$$

$$|2,-2>=\frac{3}{8\pi}e^{-i(\varphi_1+\varphi_2)}$$

And the result is

$$dP=d\theta_1 d\theta_2 d\varphi_1 d\varphi_2(\frac{3}{8\pi})^2[2+\frac{2}{3}cos^2(\varphi_1-\varphi_2)] \tag{529}$$

Or eventually

$$\frac{dP}{d\varphi}=\frac{3}{7\pi}[1+\frac{1}{3}cos^2\varphi] \tag{530}$$

3.19 q/q01j1 V

For the potential

$$V(x_1,x_2)=\frac{1}{2}M\omega^2x_1^2+\lambda\delta(x_1-x_2) \tag{531}$$

find the probability of scattering of incoming particle e^{ipx_2} with oscillator transited $0\to 1$. With

$$w=\frac{2\pi}{\hbar}\frac{1}{v}\int\frac{dk}{2\pi}\delta(E_{2i}-E_{2f}-\hbar w)\langle k,1|V_{pert}|p,0\rangle \tag{532}$$

where $|p\rangle=e^{ikx_2}$, v is the incident velocity and

$$|0\rangle=\left(\frac{2b}{\pi}\right)^{\frac{1}{4}}e^{-bx_1^2},\qquad|1\rangle=\left(\frac{2b}{\pi}\right)^{\frac{1}{4}}2\sqrt{b}x_1e^{-bx_1^2},\qquad b=\frac{M\omega}{2\hbar} \tag{533}$$

we get

$$w=\frac{2\pi m}{\hbar^3 v}\left(\frac{1}{k_1}\frac{\lambda^2q_1^2}{8b}e^{-\frac{q_1^2}{4b}}+\frac{1}{k_2}\frac{\lambda^2q_2^2}{8b}e^{-\frac{q_2^2}{4b}}\right), \tag{534}$$

where $k_{1,2}=\pm\sqrt{p^2-\frac{2m\hbar\omega}{\hbar^2}}$, and $q_i=p-k_i$.

3.20 q/q01j2 V

From

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi+a\delta(x)\psi=E\psi \tag{535}$$

it follows that the jump of the derivative is

$$\psi'|_-^+=\alpha\psi \tag{536}$$

where $\alpha=\frac{2m\alpha}{\hbar^2}$. At each band the wave function is $a_ie^{ipx}+b_ie^{-ikx}$. From the gluing condition between functions from the left a_l,b_l and the right side a_r,b_r of the delta-function inserted in the point x the following relation follows

$$\begin{pmatrix}a_l\\ b_l\end{pmatrix}=\begin{pmatrix}1-\frac{\alpha}{2ip} & -\frac{\alpha}{2ip}e^{-2ipx}\\ \frac{\alpha}{2ip}e^{2ipx} & 1+\frac{\alpha}{2ip}\end{pmatrix}\begin{pmatrix}a_r\\ b_r\end{pmatrix} \tag{537}$$

Multiplying two matrices with $x=0$ and $x=s$ we get the transition matrix M from right side of the whole potential to the left side. The transmission coefficient is

$$T=\frac{1}{|M_{11}|^2} \tag{538}$$

And

$$M_{11} = \left(1 - \frac{\alpha}{2ip}\right)^2 + \left(\frac{\alpha}{2p}\right)^2 e^{2ips}, \tag{539}$$

The reflection coefficient is proportional to M_{12}, we need to find when it vanishes. The equation is

$$(1 + b^2) + (1 - b^2)\cos 2\phi + 2b \sin 2\phi = 0 \tag{540}$$

where $b = \frac{\alpha}{2p}$ and $\phi = ps$. From it follows the solution $\phi = \delta + \frac{\pi + 2\pi k}{2}$, where $\tan\delta = \frac{\alpha}{2p}$

3.21 q/q01j3 V

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \cos\frac{gBt}{2} & i\sin\frac{gBt}{2} \\ i\sin\frac{gBt}{2} & \cos\frac{gBt}{2} \end{pmatrix} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \tag{541}$$

$$gBT = \pi \tag{542}$$

The probability to remain in the same state is $\cos^2\phi$, where $\phi = \frac{\pi}{2N}$. Thus answer for a) is

$$P_{alwaysup} = (\cos^2\phi)^N \sim 1 - \frac{\pi^2}{2N} \tag{543}$$

The probability of overturn (regardless of whether it was up or down) is $p = \sin^2\phi$. The probability of k overturns is $p^k(1-p)^{N-k}$. We need to sum up over even number of overturns.

$$\sum_{k=0,k-even}^{N} C_N^k p^k (1-p)^{N-k} = \frac{1}{2}[(1 + (1-2p)^N] \tag{544}$$

3.22 q/q01m1 V

Explicitly the Hamiltonian $H = -\mu\sigma_i B_i$ has the form

$$-\mu B \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{545}$$

with eigenvalues $-\mu B\{1, -1\}$ and the normalized eigenstates correspondingly

$$\psi^0_+(\phi) = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \psi^0_-(\phi) = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{546}$$

The B field rotates and the resulting Hamiltonian could be expressed as a result of the unitary transformation of the original Hamiltonian:

$$H(t) = U_\phi^{-1} H_0 U_\phi \tag{547}$$

where

$$H_0 = -\mu B \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{548}$$

$$U_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \tag{549}$$

That corresponds to the transformation of the eigenstates

$$\psi^0_\pm(\phi) = U_\phi^{-1}\psi^0_\pm(\phi = 0) \tag{550}$$

For the Hamiltonian which does not depend of time the evolution of the eigenstate with energy E is

$$i\hbar\dot{\psi} = H\psi, \qquad \psi(t)_{\pm} = \psi(t=0)e^{-\frac{iE_{\pm}t}{\hbar}} \tag{551}$$

Now, turn on the rotation of the Hamiltonian and try to find the solution

$$i\hbar\dot{\Psi}(t) = U^{-1}_{\phi(t)} H_0 U_{\phi(t)} \Psi(t) \tag{552}$$

in the following form

$$\Psi(t) = A(t)e^{-\frac{iE_{+}t}{\hbar}} U^{-1}_{\phi(t)} \psi^0_+ + B(t)e^{-\frac{iE_{-}t}{\hbar}} U^{-1}_{\phi(t)} \psi^0_- \tag{553}$$

Substituting in the equation of motion we get:

$$\dot{A}\psi^0_+ + \dot{B}\psi^0_- + U\dot{U}^{-1}(A\psi^0_+ + B\psi^0_-) = 0 \tag{554}$$

Evaluating the matrix elements of the operator U in the basis $\left|\psi^0_{\pm}\right\rangle$ we get the equation:

$$\begin{pmatrix} \dot{A} \\ \dot{B} \end{pmatrix} = i\dot{\phi} \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{555}$$

with the solution

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = e^{i\frac{\phi}{2}} \begin{pmatrix} \cos\theta\sin\frac{\phi}{2} + \cos\frac{\phi}{2} & \sin\theta\sin\frac{\phi}{2} \\ \sin\theta\sin\frac{\phi}{2} & -\cos\theta\sin\frac{\phi}{2} + \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} A(0) \\ B(0) \end{pmatrix} \tag{556}$$

What is strange is that we got the exact solution, not approximate one.. Where is mistake? And since $\sin\frac{\phi}{2} = 0$ at $\phi = 2\pi$ the system returns into the same state (of course with the usual time-phase factor). The probability to be excited is 0 ????

3.23 q/q01m2 V

a,b)

The atoms are neutral. They can have dipole, quadrupole,... multipole moments. The energy of the dipole in the field of another dipole

$$U(r) = \frac{3(d_1 r)(d_2 r) - r^2 d_1 d_2}{r^5} \tag{557}$$

is proportional to $d_1 d_2 / r^3$.

For the ground state the dipole moment is zero. The contribution to the energy appears from the second order of the perturbation theory.

$$\delta E = -\sum_n \frac{|V_{0n}|^2}{E_n - E_0} \tag{558}$$

To the ground state it is always negative. By dimensional reasoning $d \sim a_b e$, where a_b is obtained from $\frac{\hbar^2}{m a_b^2} = \frac{e^2}{r}$, therefore $a_b = \frac{\hbar^2}{me^2}$, and $d \sim \frac{hbar^2}{me}$, and $E \sim \frac{me^4}{\hbar^2}$. Thus

$$\delta E \sim -\left(\frac{\hbar^2}{me}\right)^4 \frac{\hbar^2}{me^4} r^{-6} \tag{559}$$

c) For the first excited state the perturbation to the energy will be given by the solution of the secular equation. Since the non-diagonal matrix elements of the dipole moment do not vanish, there will be non zero contribution already in the first order of perturbation theory.

$$\delta E \sim \frac{dd}{r^3} \tag{560}$$

d) The characteristic time of processes inside atom is the period of the emitted light. Therefore the retarding could be important at the distances larger than the wavelength of the emitted light.

$$\lambda \sim \frac{c}{\omega} \sim \frac{c\hbar^3}{me^4} \tag{561}$$

3.24 q/q01m3 V

The probability of scattering per time unit for the plane wave normalized for 1 incoming particle contained in the box $L \times L$, with the wave function $|i_L\rangle = \frac{1}{L}|i\rangle = \frac{1}{L}e^{ipx}$ and the outgoing particle with one particle per unit volume (the wave function is $|f\rangle = e^{ikx}$) is given by

$$w = \frac{2\pi}{\hbar}\int \frac{d^2k}{(2\pi)^2}\delta(E_f - E_i)\frac{1}{L^2}|\langle f|V|i\rangle|^2 \tag{562}$$

The probability of scattering for just one particle is obtained

$$p = w\frac{L}{v} = \frac{2\pi}{\hbar v}\int \frac{d^2k}{(2\pi)^2}\delta(E_f - E_i)\frac{1}{L}|\langle f|V|i\rangle|^2 = \frac{2\pi}{\hbar v}\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\int_0^{2\pi} d\theta_{q=ksin\theta}\frac{1}{L}|\langle f|V|i\rangle|^2 \tag{563}$$

$$\frac{1}{L}|\langle f|V|i\rangle|^2 = \lambda^2\frac{1}{L}\left|\lambda\frac{1}{4aN}\sum_{n=-N}^{n=N}\int_{2na}^{(2n+1)a} e^{iqx}\right|^2 = \lambda^2\frac{1}{4aN}\left|\frac{1}{iq}\sum_{n=-N}^{n=N}(e^{iqa}-1)e^{2inqa}\right|^2 = \tag{564}$$

$$= \lambda^2\frac{a^2}{4aN}\left|\frac{1}{qa}\frac{\sin 2(N+\frac{1}{2})qa}{\cos\frac{qa}{2}}\right|^2 = \tag{565}$$

$$\tag{566}$$

Using relation

$$\frac{1}{\pi A}\int_{-\infty}^{\infty}\frac{\sin^2 A\xi}{\xi^2}d\xi = 1 \tag{567}$$

we substitute at $\frac{1}{\pi A}\frac{\sin^2 A\xi}{\xi^2} = \delta(\xi)$ at $A \to \infty$ and get

$$\frac{1}{L}|\langle f|V|i\rangle|^2 = \frac{\pi a}{2}\delta(qa) + \frac{\pi a}{8}\sum_{k=-\infty}^{k=\infty}\delta(qa - \pi(2k+1)) \tag{568}$$

Note, how N has been cancelled. Therefore the probability of scattering is the sum of scatterings to discrete angles with the condition $qa = (2k+1)\pi a$ or $qa = 0$.

$$p = \lambda^2\frac{2\pi}{\hbar v}\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\int_0^{2\pi} d\theta_{q=ksin\theta}\left[\frac{\pi a}{2}\delta(qa) + \frac{\pi a}{8}\sum_{n=-\infty}^{n=\infty}\delta(qa - \pi(2n+1))\right] \tag{569}$$

Therefore the answer is

$$p = \lambda^2\frac{2\pi}{\hbar v}\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\left[\frac{\pi}{2} + \frac{\pi}{8}\sum_{|2n+1|<\frac{qa}{\pi}}\frac{1}{\sqrt{1-\left(\frac{\pi(2n+1)}{a}\right)^2}}\right] \tag{570}$$

That was the probability of scattering at all angles. If we want to compute only backward scattering then we need to take one half of the contribution from the second term (in large brackets).

3.25 q/q02m1 T

Equations of motion are

$$\dot{x}(t) = i[H, x] = p(t) \tag{571}$$

$$\dot{p}(t) = i[H, p] = -x(t) + \sqrt{2}f(t) \tag{572}$$

With condition

$$x(t=0) \equiv \hat{x}_0 \tag{573}$$

$$p(t=0) \equiv \hat{p}_0 \tag{574}$$

Thus the solution for $t<0$ is

$$x(t) = \cos t\hat{x}_0 + \sin t\hat{p}_0 \tag{575}$$

$$p(t) = \cos t\hat{p}_0 - \sin t\hat{x}_0 \tag{576}$$

For $t \in [0, T]$

$$x(t) = \cos t\hat{x}_0 + \sin t\hat{p}_0 + \frac{\sqrt{2}f_o}{(1-w^2)}[\cos wt - \cos t] \tag{577}$$

$$p(t) = \cos t\hat{p}_0 - \sin t\hat{x}_0 - \frac{\sqrt{2}f_o w}{(1-w^2)}[w\sin wt - \sin t] \tag{578}$$

And for $t > T$

$$x(t) = \cos t\hat{x}_0 + \sin t\hat{p}_0 + \frac{\sqrt{2}f_o}{(1-w^2)}A\cos(t-T+\varphi) \tag{579}$$

$$p(t) = \cos t\hat{p}_0 - \sin t\hat{x}_0 - \frac{\sqrt{2}f_o}{(1-w^2)}A\sin(t-T+\varphi) \tag{580}$$

where

$$\tan\varphi = \frac{\sin T - w\sin wT}{\cos T - \cos wT} \tag{581}$$

and

$$A = \sqrt{1+\cos^2 wT + w^2\sin^2 wT - 2(\cos wT\cos T + w\sin wT\sin T)} \tag{582}$$

c). We know that $\dot{E} = \langle 0|\frac{\partial H}{\partial t}|0\rangle$. Since at $t=0$ $\hat{x}$ has no C-valued part there is no jump in energy for $t=0$. Similarly to avoid jump in energy when we will turn perturbation of we assume that $\hat{x}$ has no C-valued part at $t=T$ as well. Using exact solution for $w=1$ this means $sinT=0$. Using this and

$$E(T) - E(0) = \frac{1}{2}\left(\frac{\sqrt{2}f_0}{(1-w^2)}\right)^2 A^2 \tag{583}$$

we have in the limit $w \to 1$

$$E(T) - E(0) = \frac{f_0^2}{4}T^2 \tag{584}$$

3.26 q/q02m2 T

q02m2 a). $x_{i\ i+1} = \frac{L}{3}$,$S=1/2$,$S=S_1+S_2+S_3$ b). Since $\alpha << 1$ then $S=1/2$ for new vacuum too. Since P_{ij} does not change S_z we will have two similar vacua for $S_z = \pm 1/2$.

Thus vacuum will have form $|a,b,c> = a|+,+,-> + b|+,-,+> + c|-,+,+>$ with $a+b+c=0$ for $S=1/2$ and $x_{12} = x_{13} = L/3+\xi$. Then hamiltonian is

$$H = J(1-\frac{\alpha L}{3})[S^2 - 9/4] + \frac{kL^2}{6} + 3k\xi^2 - \frac{\alpha\xi}{2}[P_{12}+P_{13}-2P_{23}] \tag{585}$$

For —-b-c,b,c¿ to be an eigenvector the following equation(s) should be satisfied

$$H|-b-c,b,c> = E|-b-c,b,c> \tag{586}$$

or (here $h = -\frac{3}{2}J(1-\frac{\alpha L}{3}) + k[\frac{L^2}{6}+3\xi^2]$)

$$hb + \frac{3\alpha\xi}{2}c = Eb \tag{587}$$

$$hc + \frac{3\alpha\xi}{2} b = Ec \tag{588}$$

It is important here that third equation is just the sum of first two (no additional constrains). Then $E = h \pm \frac{\alpha\xi}{2}$ (and the state is) Now we can minimize with respect to ξ:

$$\xi = \pm \frac{\alpha}{4k} \tag{589}$$

but in the both cases $a = b$ and only one eigenvector (plus the same for $S_{1/2}$). So the ground state is $|-2a, a, a>$ $, \xi = +\alpha/4k$

3.27 q/q02m3 T

q02m3 a). $A_y = 0$ for $x < 0$, $A_y = B_0 x$ for $x \in [0..d]$ and $A_y = B_0 d$ for $x > d$

$$H = -\frac{\hbar^2}{2m}[\partial_x^2 + (\partial_y + iA_y)^2] \tag{590}$$

We will restrict ourselves for the wave functions of the form $\Psi = \Psi(x)$ and thus $\partial_y = 0$

$$H = -\frac{\hbar^2}{2m}[\partial_x^2 - A_y^2] \tag{591}$$

b). When $x < 0$ and $\Psi = e^{ikx}$ we have no constrains on k. When $x > 0$ and $\Psi = e^{i\tilde{k}x}$ then we have a constraint $k^2 = \tilde{k}^2 + (B_0 d)^2$ c). Critical k is equal to $B_0 d$. Classicaly magnetic field will rotate (turn the particle and if $k < B_0 d$ then it will return back). At first $v = \frac{k}{m}$. Then $F = \frac{mv^2}{r} = B_0 v$. Thus the radius of the circle is $r = \frac{k}{B_0}$. If $d > r$ the particle will return back. d).

$$J_i = \frac{i\hbar^2}{2m}[\bar{\Psi}(\partial_i - iA_i)\Psi - (\partial_i + iA_i)\bar{\Psi}\Psi] \tag{592}$$

$x < 0$ $\Psi = e^{ikx} + RE^{-ikx}$

$x > d$ $\Psi = Te^{i\tilde{k}x}$ Then

$$J_x = \frac{i\hbar^2}{2m}k(1 - R^2), \qquad x < 0$$

$$J_x = \frac{i\hbar^2}{2m}\tilde{k}T, \qquad x > d$$

$$J_y = \frac{\hbar^2}{m}B_0 x|\Psi|^2, \qquad x \in [0..d]$$

Elsewhere $J_y = 0$ There is a flow along x-axis,as usual (we will see below that the flow to the left of 0 is equal to the flow to the right of d). And there is also probability flow along y-axis, inside the strip, filled with magnetic flux.

e). Integrating Schrodinger equation fro 0 to d and taking d to 0 we get that $\Psi(0) = \Psi(d)$ and $\partial_x\Psi(d) - \partial_x\Psi(0) \sim d \to 0$ Thus $\Psi(0) = \Psi(d) \Leftrightarrow (1 + R) = T$ and $\partial_x\Psi(d) = \partial_x\Psi(0) \Leftrightarrow k(1 - R) = \tilde{k}R$ As a result

$$T = \frac{2k}{k + \tilde{k}}, \qquad R = \frac{k - \tilde{k}}{k + \tilde{k}} \tag{593}$$

3.28 q/q02j1 T

Wave function has form

$$\Psi(x) = \frac{\psi(x)}{x} \tag{594}$$

Then

$$\psi_<(x) = \sinh(kx) \tag{595}$$

$$\psi_=(x) = \alpha e^{ipx} + \beta e^{-ipx} \tag{596}$$

and

$$\psi_>(x) = Ae^{-k(x-b)} \tag{597}$$

Now equation about smoothness of our wave-function imply

$$\alpha = \frac{1}{2p}e^{-ipx}[p\sinh ka - ik\cosh ka] = \frac{A}{2p}e^{-ipb}[p+ik] \tag{598}$$

$$\beta = \frac{1}{2p}e^{ipx}[p\sinh ka + ik\cosh ka] = \frac{A}{2p}e^{+ipb}[p-ik] \tag{599}$$

This imply that A is a real quantity or

$$\Re\frac{e^{-ip(b-a)}[p\sinh ka + ik\cosh ka]}{[p-ik]} = 0 \tag{600}$$

or

$$\tan(p(b-a)) = \frac{pk\sinh ka\cosh ka}{p^2\sinh ka - k^2\cosh ka} \tag{601}$$

We will follow RHS of the last expression as the function of p. At $p = 0$ our function is 0. LHS is also zero but this is not correct solution. Moreover-there is no such solution at all-during our consideration we divided by p and our formula works only for $p \neq 0$. Then RHS decrease and goes to minus infinity at the point where $p^2\sinh ka = k^2\cosh ka$ After that RHS decreasing from plus infinity and goes to $\frac{1}{\sqrt{2mV_0 a}}$ as $k \to 0$. There is only one possibility for this curve doesn't cross tan curve: its last point should below tan brunch and $\sqrt{2mV_0}(b-a) < \frac{\pi}{2}$ (we still below the first tan brunch).

This condition

$$\sqrt{2mV_0}a\tan\sqrt{2mV_0}(b-a) = 1 \tag{602}$$

is just the condition for groundstate to have zero energy! And our second condition

$$\sqrt{2mV_0}(b-a) < \frac{\pi}{2} \tag{603}$$

means that our wave-function has no zeroes and that it is the ground-state of the system.

Conclusion: our consideration demonstrates that maximal symmetry state without zeroes is ground-state and when ground-state energy is large than zero there is no bound-states.

3.29 q/q02j2 T

a). Plane wave $\psi = e^{ikx}$ should became $\psi = e^{i(k-mv)x}$ and general function $\psi(x)$ became $\tilde{\psi} = \psi(x)e^{-imvx}$. Now we will also consider time dependent-wave function.

If

$$i\dot{\psi} = [-\frac{\Delta}{2m} + U]\psi(x,t) \tag{604}$$

then

$$i\dot{\tilde{\psi}}(x,t) = [-\frac{\Delta}{2m} + U]\tilde{\psi}(x,t) = \psi(\dot{x},t)e^{-imvx} + \frac{mv^2}{2}\psi(x)e^{-imvx} + iv\partial\psi(x,t)e^{-imvx} \tag{605}$$

Result:

$$\tilde{\psi}(x,t) = \psi(x+vt,t)e^{-imvx - i\frac{mv^2}{2}t} \tag{606}$$

b). Obviously $P = |A|^2$ and

$$A = \langle\psi_0|\left|\psi_0(x)e^{-imvx}\right\rangle \tag{607}$$

where

$$\psi_0(x) = e^{\frac{-|x|}{a}} \tag{608}$$

the Hydrogen atom ground state. We do not care here about general coefficient. After all $A(v)$ would be normalized from the condition $A(0) = 1$.

$$P \sim \int_0^\infty \int_{\xi=-1}^{\xi+1} r^2 dr e^{\frac{-2r}{a}} e^{-imvr\xi} d\xi \sim \frac{a^3}{(1+\frac{m^2v^2a^2}{4\hbar^2})^2} \tag{609}$$

Eventually

$$A(v) = \frac{1}{(1+\frac{m^2v^2a^2}{4\hbar^2})^4} \tag{610}$$

3.30 q/q02j3 T

a). First of all we separate wave function of center of mass. Or

$$\psi_{\alpha\beta}(x_1, x_2) = \psi(\frac{x_1+x_2}{2}) \otimes \Psi_{\alpha\beta}(x_1 - x_2) \tag{611}$$

The Hamiltonian for center of mass is trivial (as for free particle with mass $2M$) and $\psi(x)$ is just the plane wave.

At the next step we separate spin and space wave function. Now we have to wave functions for spin zero $\Psi_0(x)$ and for spin one $\Psi_1(x)$. Corresponding Hamiltonians are

$$H_0 = \frac{P^2}{M} + 2\hbar^2 U_0(x) \tag{612}$$

and

$$H_0 = \frac{P^2}{M} \tag{613}$$

Eigenstates with total spin zero are

$$\Psi_0(x) = \sqrt{\frac{2}{\pi a}} \sin\frac{\pi n(x+a)}{2a}, \qquad E_n = \frac{\pi^2\hbar^2(n^1-1)}{4Ma^2} \tag{614}$$

with integer n.

Eigenstates with total spin one are

$$\Psi_{S^2=1,S_Z}(x) = \sqrt{\frac{1}{L}} e^{ikx}, \qquad E_k = \frac{k^2}{M} \tag{615}$$

with arbitrary k. The energy of the ground state is zero.

b).Since the system is in the groundstate we conclude that momentum of center of mass is zero. Then expanding external field in the length of the system we have the following perturbing Hamiltonian

$$H_1 = S_z \mu B(\frac{x_1+x_2}{2}) + \frac{x_1-x_2}{2}\mu B'(\frac{x_1+x_2}{2})[S_z^1 - S_z^2] \tag{616}$$

We have to calculate

$$\left\langle \Psi_{S^2=1,S_z}(k) \right| H_1 \left| \Psi_0(n=0) \right\rangle \tag{617}$$

Since $S_z\Psi_0 = 0$ we drop the first term of H_1. We also express $\cos(kx - wt)$ as $\Re e^{i(kx-wt)}$ Then we have to calculate

$$P = \delta_{S_z=1} \sqrt{\frac{2}{\pi a L}} \int_{-a}^{a} \sin(\frac{\pi(x+a)}{2a}) \frac{1}{2} e^{ikx} \mu k B_0 \tag{618}$$

dx All this could be multiplied by phase factor from $B(x) \sim e^{ikx}$ in the center of mass. Calculating this one gets

$$P = \delta_{S_z=1} \sqrt{\frac{1}{8\pi a L}} \mu B_0 [\frac{\cos(ka)}{ka+\frac{\pi}{2}} - \frac{\cos(ka)}{ka-\frac{\pi}{2}}] = \delta_{S_z=1} \sqrt{\frac{1}{8\pi a L}} \mu B_0 \frac{4\pi \cos ka}{\pi^2 - (2ka)^2} \tag{619}$$

Now probability per second is

$$T^{-1} = \frac{4\pi^2 M\mu^2 B_0^2 \cos^2(ka)}{k(\pi^2 - (2ka)^2)^2} \tag{620}$$

Halftime T_2 is $T \log 2$.

3.31 q/q03m1 T

a). Wave function

$$\hat{\psi} = exp(-i\phi(t)\frac{\sigma_z}{2})\psi(t) \tag{621}$$

evolves according to the Hamiltonian

$$H_{rot} = [\frac{1}{2}\dot{\psi} + \mu B_0]\sigma_z + exp(-i\phi(t)\frac{\sigma_z}{2})[\sigma_x \cos\phi + \sigma_y \sin\phi]exp(i\phi(t)\frac{\sigma_z}{2}) \tag{622}$$

Using that

$$e^{-i\phi(t)\frac{\sigma_z}{2}}[\sigma_x \cos\phi + \sigma_y \sin\phi]e^{i\phi(t)\frac{\sigma_z}{2}} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{623}$$

And eventually

$$H_{rot}(t) = \begin{pmatrix} \mu B_0 + \frac{1}{2}\dot{\phi} & \mu B_1 \\ \mu B_1 & -(\mu B_0 + \frac{1}{2}\dot{\phi}) \end{pmatrix} \tag{624}$$

In the case $\ddot{\phi} = 0$ H_{rot} doesn't depend on t.

b). $\phi = w_1 t$. Then H_{rot} has eigenvalues

$$\pm\sqrt{\mu^2 B_1^2 + (\mu B_0 + \frac{1}{2}w_1)^2} \tag{625}$$

with eigenvectors (not normalized)

$$\begin{pmatrix} 1 \\ \frac{\sqrt{\mu^2B_1^2+(\mu B_0+\frac{1}{2}w_1)^2}-(\mu B_0+\frac{1}{2}w_1)}{\mu B_1} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{-\sqrt{\mu^2B_1^2+(\mu B_0+\frac{1}{2}w_1)^2}-(\mu B_0+\frac{1}{2}w_1)}{\mu B_1} \end{pmatrix} \tag{626}$$

correspondingly.

Since $e^{-i\frac{\phi\sigma_z}{2}}$ preserve up-down directions we can consider $\hat{\psi}$ instead of ψ studying spin flipping. At the moment $t = -T$ spin was directed down. Time-dependent wave-function is

$$w \equiv \sqrt{\mu^2 B_1^2 + (\mu B_0 + \frac{1}{2}w_1)^2} \tag{627}$$

$$\hat{\psi}(t) = e^{-iw(t+T)} \begin{pmatrix} 1 \\ \frac{\sqrt{\mu^2B_1^2+(\mu B_0+\frac{1}{2}w_1)^2}-(\mu B_0+\frac{1}{2}w_1)}{\mu B_1} \end{pmatrix} + e^{iw(t+T)} \begin{pmatrix} -1 \\ \frac{+\sqrt{\mu^2B_1^2+(\mu B_0+\frac{1}{2}w_1)^2}+(\mu B_0+\frac{1}{2}w_1)}{\mu B_1} \end{pmatrix} \tag{628}$$

Condition that at the moment $t = T$ spin will be directed up is

$$-\cot(2wT) = \frac{\sqrt{w^2 - \mu^2 B_1^2}}{w} \tag{629}$$

c). Ground level has energy $-w(t)$. At $t = -T$ corresponding vector has form

$$\begin{pmatrix} 1 \\ \frac{\mu B_1}{\alpha T - 2\mu B_0} + O(\frac{1}{(\alpha T)^2}) \end{pmatrix} \tag{630}$$

spin is up.

At $t = +T$ corresponding vector has form

$$\begin{pmatrix} 1 \\ -\frac{\alpha T}{\mu B_1} + O(1) \end{pmatrix} \tag{631}$$

spin is down.

3.32 q/q03m2 T

a). Energy of bound state is $E = -\frac{k^2}{2m}$. Then equation which determines these states is

$$\tan\phi = -\frac{-\phi}{\sqrt{\phi_0^2 - \phi^2}} \tag{632}$$

Here $\phi = r_0\sqrt{2mV_0 - k^2}$ and $\phi_0 = \phi(k=0)$. These equation has at least one solution if RHS well defined to the right of $\phi = \frac{\pi}{2}$. In the opposite case there is no solution. Thus there is no solution if $\phi(k=0) = \frac{\pi}{2}$ or

$$V_{cr} = \frac{\pi^2}{8mr_0^2} \tag{633}$$

This condition coincides with the condition $k = 0$.

b).By the definition δ_l is the shift from the formula

$$R_{kl} \to \sin(kr - \frac{\pi l}{2} + \delta_l(k)) \tag{634}$$

For $l = 0$ equation for δ_l is

$$\frac{\tan(\sqrt{2mV_0 + k^2}r_0)}{\sqrt{2mV_0 + k^2}} = \frac{\tan(kr_0 + \delta(k))}{k} \tag{635}$$

If $k \to 0$ then

$$\delta(k) = k[\frac{\tan(\sqrt{2mV_0}r_0)}{\sqrt{2mV_0}} - r_0] \tag{636}$$

c). A obviously vanishes when $V_0 \to 0$ and goes to infinity when $V_0 \to V_{cr}$. d).

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k) \tag{637}$$

In the case $k \to 0$

$$\sigma_l = 4\pi[\frac{\tan(\sqrt{2mV_0}r_0)}{\sqrt{2mV_0}} - r_0]^2 \tag{638}$$

When $V \to V_{cr}$ $\delta(k)$ is not small any more. Thus

$$\sigma_l \sim \frac{1}{k^2} = \frac{1}{E - E_0} \tag{639}$$

or there is pole in cross -section.

3.33 q/q03m3 T

a). Representation $J = 3/2$

$$|J = 3/2, J_z = 3/2\rangle = |1\rangle \otimes |1/2\rangle \tag{640}$$

$$|J = 3/2, J_z = 1/2\rangle = \frac{1}{\sqrt{3}}[\sqrt{2}\,|0\rangle \otimes |1/2\rangle + |1\rangle \otimes |-1/2\rangle] \tag{641}$$

$$|J = 3/2, J_z = -1/2\rangle = \frac{1}{\sqrt{3}}[\sqrt{2}\,|0\rangle \otimes |-1/2\rangle + |-1\rangle \otimes |1/2\rangle] \tag{642}$$

$$|J = 3/2, J_z = -3/2\rangle = |-1\rangle \otimes |-1/2\rangle \tag{643}$$

Thus

$$\langle J = 3/2, J_z|\, S_z\, |J = 3/2, J_z\rangle = \frac{1}{3}J_z \tag{644}$$

Representation $J = 1/2$

$$|J = 1/2, J_z = 1/2\rangle = \frac{1}{\sqrt{3}}[\sqrt{2}\,|1\rangle \otimes |-1/2\rangle - |0\rangle \otimes |1/2\rangle] \tag{645}$$

$$|J = 1/2, J_z = -1/2\rangle = \frac{1}{\sqrt{3}}[\sqrt{2}\,|-1\rangle \otimes |1/2\rangle - |0\rangle \otimes |-1/2\rangle] \tag{646}$$

Thus

$$\langle J = 1/2, J_z|\, S_z\, |J = 1/2, J_z\rangle = -\frac{1}{3}J_z \tag{647}$$

b). Assuming that

$$\langle J, J_z|\, S_z\, |J, J_z\rangle = g_J J_z \tag{648}$$

we can calculate g_J for only one J_z.

For $J = l + 1/2$ we take $|J = l + 1/2, J_z = l + 1/2\rangle = |l\rangle \otimes |1/2\rangle$
and thus

$$\langle l + 1/2, l + 1/2|\, S_z\, |l + 1/2, l + 1/2\rangle = 1/2 \tag{649}$$

Conclusion

$$g_{l+1/2} = \frac{1}{1 + 2l} \tag{650}$$

For $J = l - 1/2$ we will also take vector with $J_z = J$. In order not to write down it explicitly we note that $[S_z, J_z]$. Let us consider all (two) vectors with $J_z = l - 1/2$. They are

$$|a\rangle = |l + 1/2, l - 1/2\rangle \tag{651}$$

and

$$|b\rangle = |l - 1/2, l - 1/2\rangle \tag{652}$$

Now

$$S_z\,|a\rangle = \frac{2l - 1}{2(2l + 1)}\,|a\rangle + xb \tag{653}$$

since $\langle a|\,|b\rangle = 0$ and $\langle a|\, S_z\, |a\rangle = \frac{2l-1}{2(2l+1)}$. Let us act by S_z to $S_z\,|a\rangle$ once again. Using that $S_z^2 = 1/4$ (property of $S^2 = 3/2$ representation) we have

$$\frac{1}{4}[1 - \left(\frac{2l - 1}{(2l + 1)}\right)^2]\,|a\rangle = xS_z b + x\frac{2l - 1}{2(2l + 1)}\,|b\rangle \tag{654}$$

Now using orthogonality one gets

$$\langle b|\, S_z\, |b\rangle = -\frac{2l - 1}{2(2l + 1)} \tag{655}$$

or

$$g_{l-1/2} = -\frac{1}{1 + 2l} \tag{656}$$

c). Here $\overrightarrow{A} = \overrightarrow{S}$ and

$$\overrightarrow{A}\,\overrightarrow{S} = S^2 + \frac{1}{2}[J^2 - L^2 - S^2] = \frac{1}{2}[J^2 + S^2 - L^2] \tag{657}$$

For $J = l + 1/2$ it is $\overrightarrow{A}\,\overrightarrow{S} = \frac{1}{2}(l + 3/2)$ And $J(J + 1) = \frac{1}{2}(2l + 1)(l + 3/2)$.

For $J = l - 1/2$ it is $\overrightarrow{A}\overrightarrow{S} = -\frac{1}{2}(l - 1/2)$ And $J(J + 1) = -\frac{1}{2}(2l + 1)(l - 1/2)$.

4 Statistical Physics

4.1 s/s98j1 V

We consider that the massless quarks are Fermi-distribute inside the confined volume of a ball with the radius R.

Tnen, if the degeneracy factor is $b = 18$ we get from the $\mu = 0$ Fermi distribution (using $d^3p = d\Omega E^2 dE$)

$$N = \frac{4}{3}\pi R^3 b \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty \frac{E^2 dE}{e^{E/T}+1} = \frac{4}{3}\pi R^3 b \frac{4\pi}{(2\pi\hbar)^3} T^3 c_1 \tag{658}$$

$$E/2 = \frac{4}{3}\pi R^3 b \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty \frac{E^3 dE}{e^{E/T}+1} = \frac{4}{3}\pi R^3 b \frac{4\pi}{(2\pi\hbar)^3} T^4 c_2 \tag{659}$$

where

$$c_1 = \frac{3}{2}\zeta(2), \qquad c_2 = \frac{7}{120}\pi^4 \tag{660}$$

And

$$N = \left(\frac{ER}{\hbar c}\right)^{3/4} \left(\frac{b}{24\pi}\right)^{\frac{1}{4}} a_1 a_2^{-\frac{3}{4}} \tag{661}$$

4.2 s/s98j2 V

We consider that $\mu = -\lambda$

From the grand canonical distribution

$$w_{nN} = e^{\frac{\Omega + \mu N - E_{nN}}{T}} \tag{662}$$

in the limit $T \gg |\mu|$, when $e^{\mu/T} \ll 1$ we have (Boltzman statistics)

$$\Omega = -T\sum_k \log(1 - e^{\frac{\mu - E_k}{T}}) \sim T\sum_k e^{\frac{\mu - E_k}{T}} \tag{663}$$

For classical non interacting gas in the box sum the sum is readily computed with

$$PV = -\Omega = e^{\mu/T} T \sum e^{-\frac{E_k}{T}} = e^{\mu/T} T \int \frac{d^3p d^3x}{(2\pi\hbar)^3} e^{-\frac{E(p)}{T}} = V T^{5/2} e^{\mu/T}\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \tag{664}$$

Therefore a)

$$P = T^{5/2} e^{\mu/T}\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \tag{665}$$

b)

$$\mu = T\log\left(PT^{-5/2}\left(\frac{2\pi\hbar^2}{m}\right)\right) \tag{666}$$

c)

$$\dot{Q} = l\dot{N} = \lambda\frac{\dot{V}P}{T} \tag{667}$$

4.3 s/s98j3 V

a,b) The partition function is

$$Z = (1 + e^{-E/T})^N \tag{668}$$

where $E = g\mu_b \hbar H$. Then

$$F = -T \log Z, \qquad S = -\frac{dF}{dT} \tag{669}$$

$$S = N\left(\log(1 + e^{-E/T}) + \frac{E}{T}\frac{e^{-E/T}}{1 + e^{-E/T}}\right) \tag{670}$$

The entropy $S = 0$ at $T = 0$ and $S = \log 2^N$ at $T = \infty$ as it should be.

c) When the external field increase the system tends more to occupy ground state, therefore it emits heat.

$$\delta Q = T(S_f - S_i) \tag{671}$$

d) In the adiabatic process S is conserved, therefore

$$H_f/T_f = H_i/T_i \tag{672}$$

4.4 s/s98m1 V

For the ideal gas with constant c_v the adiabatic process takes the form $pV^\gamma = \text{const}$, where $\gamma = \frac{c_p}{c_v}$. The isotermic follows from the equation of state $pV = \nu RT$. The work is

$$W = \oint p dV = p_1 V_1 \log\frac{V_2}{V_1} + p_2 V_2 \frac{1}{\gamma - 1}\left(1 - \left(\frac{V_2}{V_3}\right)^{\gamma-1}\right) - p_3 V_3 \log\frac{V_3}{V_4} - p_1 V_1 \frac{1}{\gamma - 1}\left(1 - \left(\frac{V_1}{V_4}\right)^{\gamma-1}\right) \tag{673}$$

Since $p_3 V_3 = p_2 V_2 \left(\frac{V_2}{V_3}\right)^{\gamma-1}$, and $p_1 V_1 = p_2 V_2 = \nu R T_1$, and $p_3 V_3 = p_4 V_4 = \nu R T_2$ we get the answer

$$T_1 = \frac{W}{\nu R}\left[\left(1 - \left(\frac{V_2}{V_3}\right)^{\gamma-1}\right)\log\frac{V_3}{V_4}\right]^{-1} \tag{674}$$

$$T_2 = T_1 \left(\frac{V_2}{V_3}\right)^{\gamma-1} \tag{675}$$

and the consistency condition $\frac{V_2}{V_3} = \frac{V_1}{V_4}$ (therefore one value of volume is unnecessary to specify).

4.5 s/s98m2 V

To check whether the bose-condensation takes place or not count the possible number of particles on the excited levels with $\mu = 0$.

$$N = \frac{V}{(2\pi\hbar)^3}\int \frac{d^3p}{e^{\frac{E}{T}} - 1} \tag{676}$$

with

$$d^3p = 4\pi\sqrt{2mE} m dE \tag{677}$$

and

$$\int_0^\infty \frac{x^{\alpha-1}dx}{e^x - 1} = \Gamma(\alpha)\zeta(a) = \xi(\alpha) \tag{678}$$

equals

$$N = \left(\frac{L\sqrt{mT}}{2\pi\hbar}\right)^3 4\sqrt{2}\pi\xi(3/2) = 1.2\cdot 10^{17} \tag{679}$$

Since it is less than the given number of particles, bose condensation takes place, and the ground level is occupied by macroscopic number of particles. Since in this approximation $|\mu| \ll |E_1 - E_0|$, the number of particles on the first excited energy level with $g = 3$ and $\Delta E_1 = E_1 - E_0 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(2^2-1)$ is given

$$\bar{n}_1 = \frac{g}{e^{\frac{E_1-E_0}{T}} - 1} = 8\left(\frac{L\sqrt{mT}}{2\pi\hbar}\right)^2 = 1.8\cdot 10^{11} \tag{680}$$

4.6 s/s98m3 V

The limit $T \to 0$ is the limit of $\beta = \frac{1}{T} \to \infty$, when the perturbation theory for computing $Z = \mathrm{Tr}\, e^{-\beta H}$ for approximately quadratic hamiltonian works.

$$Z = \int \frac{dpdx}{2\pi\hbar} e^{-\beta H(p,x)} = \frac{1}{2\pi\hbar}\int dp e^{-\beta\frac{p^2}{2m}} \int dx e^{-\beta(cx^2+gx^3+fx^4)} = \sqrt{\frac{\pi^2 2m}{\beta^2 c}}\left(1 - \frac{3}{4}\frac{1}{\beta}\frac{f}{c^2} + \frac{15}{16}\frac{g^2}{\beta c^3} + O(\beta^{-2})\right) \tag{681}$$

$$Z = \mathrm{const}\cdot T\left(1 - \frac{3}{4}\frac{f}{c^2}T + \frac{15}{16}\frac{g^2}{c^3}T + O(T^2)\right) \tag{682}$$

$$c = \frac{dE}{dT} = T\frac{dS}{dT} = T\frac{d}{dT}\left(-\frac{dF}{dT}\right), \quad F = -T\log Z \tag{683}$$

therefore

$$c = 1 - \frac{3f}{2c^2}T + \frac{15}{8}\frac{g^2}{c^3}T + O(T^2) \tag{684}$$

4.7 s/s99j1 T

s99j1 a).

$$< n_\epsilon >= [e^x - 1]^{-1} - 3[e^{3x} - 1]^{-1}, \qquad x = (\epsilon - \mu)/\tau \tag{685}$$

$(\mu - \epsilon)/\tau = -\infty \qquad < n_\epsilon = 0 >$
$(\mu - \epsilon)/\tau = 0 \qquad < n_\epsilon = 1 >$
$(\mu - \epsilon)/\tau = +\infty \qquad < n_\epsilon = 2 >$
b). $dN = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}}dE$
c). $\mu = \frac{\pi^2\rho^2\hbar^2}{8m}, \qquad \rho = N/L$
d). $< E >= N\mu/3$ Substitute μ from c).
e). $C_V = \frac{NT}{2\mu(T=0)}\int_{-\infty}^{\infty} dx x < n_\epsilon > (x)$ and $\alpha = -1$

4.8 s/s99j2 T

a). Since $P = -\frac{\partial F}{\partial V}$ and $F = Vf$ we have

$$P = -(1 - \frac{\partial}{\partial \log\rho})f(T,\rho) \tag{686}$$

b). Obviously

$$\frac{\partial P}{\partial \rho} = \rho f'' > 0 \tag{687}$$

Or $f'' > 0$.

c).(In this process we start from very small ρ and increase it during speculations.) We can decrease f if changing it into a straight line! Really, at first $f'' > 0$ and then $f'' < 0$. And the straight line will be below f curve. That straight line means that instead of pure gas with given $f(T, \rho)$ there are gas and liquid (condensed from this gas). Now we will construct $\tilde{f}$.

At the point ρ_g we understand that gas became into gas plus liquid. In this case $f_g = f(\rho_g)$ (the corresponding pressure is the pressure of liquid with given temperature) and do not increases at all. All additional matter (gas) became liquid. Its free energy per volume we denote as f_l assuming that this quantity corresponds to the given pressure of the gas $P_l = P_g(\rho_g)$ or

$$-P = f_g - \rho_g f'(\rho_g) = f_l - \rho_l f'(\rho_l) \tag{688}$$

Here the density of liquid is ρ_l. This equation determines ρ_l as a function of ρ_g. (* See below)

We apply this conditions by hands from the physical sense but we can change it by the condition that $\tilde{f}$ we construct should be smooth! Now we construct $\tilde{f}$ and can do anything. After construction we will check or result.

So, we proceed with construction $\tilde{f}$ and will check this condition afterwards.

Now we have the following equations for volumes

$$V_g + V_l = V \tag{689}$$

and

$$V_g \rho_g + V_l \rho_l = V\rho \tag{690}$$

Finally

$$\tilde{f} = \frac{V_g}{V} f_g + \frac{V_l}{V} f_l = \frac{\rho(f_l - f_g) + f_g\rho_l - f_l\rho_g}{\rho_l - \rho_g} \tag{691}$$

for $\rho > \rho_g$ and $\tilde{f} = f$ for $\rho < \rho_g$. Obviously $\tilde{f}'' = 0$ and this is better than $f'' > 0$ at first and $f'' < 0$ than.

This formulas works well until $\tilde{f} < f$. At the point ρ_1 where $\tilde{f} = f$ we have to switch to f again.

We understand that this switching should occur when all our matter will be liquid. And now we will demonstrate that this is so. Our equation is (fist check of smoothness $\tilde{f}$)

$$\tilde{f}(\rho_1) = f(\rho_1) \tag{692}$$

(*)Using that our liquid is the phase of our gas it should also be described by our curve. We conclude that $f_l = f(\rho_l)$. Now it is easy to check that $\rho_1 = \rho_l$ is the solution and

$$\tilde{f}(\rho_l) = f(\rho_l) \equiv f_l \tag{693}$$

Once again, we construct $\tilde{f}$ for $\rho_g < \rho < \rho_l$ and one could simply check that $\tilde{f}(\rho_g) = f(\rho_g)$ and $\tilde{f}(\rho_l) = f(\rho_l)$. Only one ambiguity we do not know precise value of ρ_g.

So the equation to determine ρ_g is our second check: $\tilde{f}'(\rho_g) = f'(\rho_g)$ and $\tilde{f}'(\rho_l) = f'(\rho_l)$. We can not solve this equation explicitly but generally it specifies ρ_g. Now one can say that we have two equation to specify one variable ρ_g, but using equation for $\rho_l(\rho_g)$

$$-P = f_g - \rho_g f'(\rho_g) = f_g - \rho_g f'(\rho_g) \tag{694}$$

we leave only one of it.

We end with construction of f and now going to pressure.

d). Now the speculations about the pressure are trivial: $P = \rho f' - f$ and if $\tilde{f}$ is smooth (with its first derivative) then P is also smooth. And it is obvious that for $\rho_l > \rho > \rho_g$ pressure is constant. So we change the curve with local minimum and maximum by the horizontal line.

4.9 s/s99j3 T

s99j3

Efficiency of refrigerator is equal to the ratio of the temperatures of refrigerated stuff and the medium outside

$$\frac{A}{Q} = \frac{T_{medium}}{T_{r.stuff}} \tag{695}$$

A is a work spent by the machine, Q is the heat taken from refrigerated stuff. T_{medium} could be arbitrary, but large than room temperature-the room (medium) should refrigerate(!) the machine. To minimize A T_{medium} should be equal to room's temperature. During the process

$$dQ = -mcdT \tag{696}$$

and

$$A = \frac{T_{medium}}{T} dQ \tag{697}$$

Integrating this equation one gets

$$A = mcT_{room} ln(T_{room}/T_{ice}) \tag{698}$$

And at the end heat to turn the water to ice is $m\lambda$ when the efficiency is T_{room}/T_{ice} . Finally

$$A = mT_{room}[\frac{\lambda}{T_{ice}} + c\ ln(\frac{T_{room}}{T_{ice}})] \tag{699}$$

4.10 s/s99m1 V

The first step is to find the equation of adiabatic process in the T, V coordinates:

$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{\partial(T,S)}{\partial(V,S)} = \frac{\partial(P,V)}{\frac{\nu C_V}{T}\partial(V,T)} = -\frac{T}{\nu C_V}\left(\frac{\partial P}{\partial T}\right)_V \tag{700}$$

In the problem $C_V = \text{const}$, $p = \frac{nRT}{V}(1+\frac{\nu\alpha}{V})$, thus at $S = \text{const}$ process

$$\int \nu C_V \frac{dT}{T} = -\int \frac{nR}{V}(1+\frac{\nu\alpha}{V})dV \tag{701}$$

$$\nu C_V \log\frac{T_f}{T_i} = -nR\left(\log\frac{V_f}{V_i} + \nu\alpha\left(\frac{1}{V_i} - \frac{1}{V_f}\right)\right) \tag{702}$$

If denote $1,2,3,4$ point on the PV diagram going clockwise from the left upper corner, then

$$T_2 = T_1\beta \tag{703}$$

$$T_3 = T_4\beta \tag{704}$$

where

$$\beta = \exp\left(-\frac{R}{C_V}\left(\log\frac{V_f}{V_i} + \nu\alpha\left(\frac{1}{V_i} - \frac{1}{V_f}\right)\right)\right) \tag{705}$$

and, since $C_V = \text{const}$ the energy of gas $U = \nu C_v T + f(V)$, where $f(V)$ does not need to be determined

$$\eta = \frac{A}{Q} = \frac{T_1 - T_2 + T_3 - T_4}{T_1 - T_4} = 1 - \beta \tag{706}$$

4.11 s/s99m2 V

Since all states have equal energy the thermodynamics of this system is governed just by the entropy $S(L)$, i.e. log number of states for a given L. Let l be length in the a units $l = L/a$. Then $N_+ = \frac{1}{2}(N+l)$, $N_- = \frac{1}{2}(N-l)$.

$$S = \log C_N^{N+} = \log N! - \log(((N-l)/2)!) - \log(((N+l)/2)!) \tag{707}$$

At $x \to \infty$

$$\log x! \sim x(\log x - 1) \tag{708}$$

therefore

$$S \sim N \log 2 - \frac{l^2}{N} = \text{const} - \frac{L^2}{Na^2} \tag{709}$$

Then, from

$$dE = TdS - FdL \tag{710}$$

and $dE = 0$ we get

$$F = T\frac{dS}{dL} = -2T\frac{L}{Na^2} \tag{711}$$

The required work to stretch from 0 to L_{max} is is

$$W = -A = T\frac{L_{max}^2}{Na^2} \tag{712}$$

With stretching the entropy decreases, and therefore the rubber gives out the heat $(0 > Q = TdS = A)$.

4.12 s/s99m3 V

The excitations $\omega^2 = \frac{\gamma}{\rho}k^3$ obey Bose statistics with $\mu = 0$. Therefore the density of the energy per area

$$E = \frac{1}{(2\pi\hbar)^2}\int \frac{d^2p\,\varepsilon}{e^{\frac{\varepsilon}{T}} - 1} \tag{713}$$

with $d^2p = \pi\hbar^2 d(k^2) = \pi\hbar^2 d(\frac{\rho}{\gamma}(\frac{\varepsilon}{\hbar})^2)^{\frac{2}{3}} = \pi\hbar^2\left(\frac{\rho}{\gamma\hbar^2}\right)^{\frac{2}{3}} d\varepsilon^{\frac{4}{3}}$ is equal to

$$E = \frac{1}{(2\pi\hbar)^2}\pi\hbar^2\left(\frac{\rho}{\gamma\hbar^2}\right)^{\frac{2}{3}}\frac{4}{3}T^{\frac{7}{3}}I \tag{714}$$

where

$$I = \int_0^\infty \frac{dx\, x^{\frac{7}{3}-1}}{e^x - 1} = \Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right) \tag{715}$$

and

$$c_V = \frac{dE}{dT} = \frac{7}{9}\Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right)\frac{1}{\pi}\left(\frac{\rho}{\gamma\hbar^2}\right)^{\frac{2}{3}}T^{\frac{4}{3}} \tag{716}$$

4.13 s/s00j1 T

Using ideal gas approximation

$$Z = \int d^3n \sum_N e^{\frac{-N(E_n-\mu)}{T}} 2^N csh^N(g\mu_b H) \tag{717}$$

Using ideal gas approximation this will be just

$$Z = \int d^3n(1 + e^{\frac{-(E_n-\mu)}{T}} 2csh(g\mu_b H) \tag{718}$$

and

$$lnZ = \int d^3n e^{\frac{-(E_n-\mu)}{T}} 2csh(g\mu_b H)) = 2csh(g\mu_b H))lnZ_0 \tag{719}$$

Pressure is given by

$$P = \frac{\partial F}{\partial V} = -\frac{\partial T lnZ}{\partial V} = P_0 2csh(g\mu_b H)) \tag{720}$$

Thus

$$\frac{P(H_1)}{P(H_2)} = \frac{csh(g\mu_b H_1)}{csh(g\mu_b H_1)} = \frac{Z_H(1)}{Z_H(2)} \tag{721}$$

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4.14 s/s00j2 T

We deal with adiabatic expansion (inverse) of ideal gas. Second low of thermodynamics says

$$dQ = 0 = dU + PdV \tag{722}$$

and

$$dU = \frac{J}{2} NkdT \tag{723}$$

where J is the number degrees of freedom. Then using following relation for ideal gas

$$PV = NkT \tag{724}$$

one simply gets

$$PV^\gamma = Const \tag{725}$$

where $\gamma = \frac{J+2}{J} = n$ Thus for N_2 $n = \frac{7}{5}$ since $J = 5$.

4.15 s/s00j3 T

a). When $\rho < rho_c r$ then $\mu < 0$ and determined from the equation

$$N = \int d\nu(E) < n(E) >= \int d^3n \frac{1}{e^{\frac{-E(n)+\mu}{T}} - 1} \tag{726}$$

$E(n) = \frac{k^2}{2m}, 2\pi n = kL$ b). When $\rho > \rho_c r$ $\mu = 0$. Distribution of particles at any excited state is given by formula from a. with $\mu = 0$. This time $N = N(T)$-number of particles at the excited states and all other particles (N-N(T)) at the ground level.

c). $N_c r$ is just the number of particles equal to $N(T)$ from a. It is corresponds to the case, when $\mu = 0$, but still microscopic number of particles at the ground level. It is easy to calculate $N_c r(T, V)$:

$$N_c r(T, V) = VT^{3/2} m^{3/2} \frac{A}{2^{1/2}\pi^2} \tag{727}$$

$$A = \int_0^\infty dt \frac{t^{1/2}}{e^t - 1} \tag{728}$$

So $\rho_c r \sim T^{3/2}$ or $\gamma = 3/2$. d). For system remains in the condensed phase A should be large enough: decreasing of temperature should be faster then decreasing of density (number of particles) for density to satisfy $\rho < \rho_c r(T)$. We will find now critical value for A when to remain in the condensed phase is still possible.

So we assume that system is in condensed phase, but without any additional funds: any time ρ is just equal to $\rho_c r$. Since volume of the system is constant this means

$$\frac{dN}{N} = \frac{3dT}{2T} \tag{729}$$

The energy of the system at critical point is just

$$E = VT^{5/2} m^{3/2} \frac{B}{2^{1/2}\pi^2} \tag{730}$$

where

$$B = \int_0^\infty dt \frac{t^{3/2}}{e^t - 1} \tag{731}$$

Number of particles, which go away from the system is $-dN > 0$. They carry out the energy $\frac{AE}{N}|dN|$ (here is no difference between mean energy of the particles before same part go away or before that-they carry small amount of energy $\sim dN$ thus difference before or after will be $\sim dN^2$). According to the relation $E = E_c r(T, N)$

$$dE = \frac{\mathcal{A} E dN}{N} = E5dT/2T \tag{732}$$

Thus

$$\frac{\mathcal{A} dN}{N} = 5dT/2T, \qquad \frac{dN}{N} = \frac{3dT}{2T} \tag{733}$$

Answer:

$$\mathcal{A} > \frac{5}{3} \tag{734}$$

4.16 s/s00m1 V

When two phases are present in the system at given temperature T the concentrations of components in them are determined by crossing of the horizontal line T with the curves separating phase areas. In the given problem, while both phases are present in the system, the concentration of x in the liquid is always three times less than that in the gas (during boiling the concentration of A in the liquid decreases). From conservation of the total amount of A:

$$x_i = (1 - p)x_f + 3px_f \tag{735}$$

where p is the part of liquid that was turned into gas, and $x_i = x, x_f = \frac{1}{2}$.

$$p = \frac{x_i - x_f}{2x_f} = \frac{1}{2} \tag{736}$$

4.17 s/s00m2 T

a).The Brownian particle started at $t = 0$ with zero initial speed from the point $r = 0$.

$$\dot{r}(t) = \frac{1}{M} \int_0^t d\xi F(\xi) - 6\pi\eta b r(t) \tag{737}$$

Let

$$\mathcal{F}(t) = \frac{1}{M} \int_0^t d\xi F(\xi) \tag{738}$$

Then

$$r(t) = \int dt' G(t,t')\mathcal{F}(t') \tag{739}$$

$$\dot{G} + 6\pi\eta b G = \delta(t-t') \tag{740}$$

and

$$G(t-t') = \theta(t-t')e^{-6\pi\eta b(t-t')} \tag{741}$$

As a result

$$r(t) = \int_0^t dt' e^{-6\pi\eta b(t-t')}\mathcal{F}(t') \tag{742}$$

To make this result easy we can integrate by parts over t' noting that $(e^x)' = e^x$ Then

$$r(t) = \frac{1}{6\pi\eta b}[\mathcal{F}(t) - \int_0^t dt' e^{-6\pi\eta b(t-t')}\frac{F(t')}{M}] \tag{743}$$

Now it is simple to calculate $\dot{r}$

$$\dot{r}(t) = \int_0^t dt' e^{-6\pi\eta b(t-t')}\frac{F(t')}{M} \tag{744}$$

b). We know that in equilibrium at temperature T every particle (non only elementary) should have energy $\frac{3}{2}kT$ or

$$\frac{3}{2}kT = \frac{M}{2} < \dot{r}^2(t) >= \frac{C}{24M\pi\eta b}[1 - e^{-12\pi\eta bt}] \tag{745}$$

Or calculations show us that energy, indeed, doesn't change with time short while after beginning of the process. Thus

$$C = 48MkT\eta\pi b \tag{746}$$

c).

$$< r^2(t) >= \frac{C}{(6\pi\eta bM)^2}[t + \frac{1}{12\pi\eta b}(-3 - e^{-12\pi\eta bt} + 4e^{-6\pi\eta bt})] \tag{747}$$

or after several mean relax times $\tau = (6\pi\eta b)^{-1}$

$$< r^2(t) >= \frac{8kT\tau}{M}[t - \frac{\tau}{4}] \tag{748}$$

k could be measured through the angle in the graph $< r^2 >$ v.s. t.

Another solution(V)

From the equation of motion for the particle

$$\dot{v} + \lambda v = f, \tag{749}$$

where $v = \dot{x}, \lambda = \frac{b}{M}, f = \frac{F}{M}$ follows the solution for the response to the $f = \delta(t)$

$$v(t) = e^{-\lambda t}, t > 0; \qquad x(t) = \frac{1}{\lambda}(1 - e^{-\lambda t}) \tag{750}$$

and by the linearity for the general $f(t)$:[1]

$$v(t) = \int_0^t d\tau e^{-\lambda(t-\tau)} f(\tau) \qquad x(t) = \int_0^t d\tau \frac{1}{\lambda}(1 - e^{-\lambda(t-\tau)}) f(\tau) \tag{752}$$

[1] Note, that the second formula could be also obtained directly from the first by changing order of integration:

$$x(t) = \int_0^t dt' \int_0^{t'} f(\tau)e^{-\lambda(t'-\tau)}d\tau = \int_0^t d\tau \int_\tau^t dt' f(\tau)e^{-\lambda(t'-\tau)} = \int_0^t d\tau f(\tau)\frac{1}{\lambda}(1 - e^{-\lambda(t'-\tau)}) \tag{751}$$

In Fourie decomposition, where $x_\omega = \frac{1}{2\pi}\int dt e^{-i\omega t}x(t)$ and $x(t) = \int d\omega x_\omega e^{i\omega t}$

$$v_\omega = \frac{f_\omega}{i\omega + \lambda} \tag{753}$$

and since $\langle f(t)f(t')\rangle = c\delta(t-t')$ with $c = \frac{C}{M^2}$

$$\langle f_\omega f_{\omega'}\rangle = \frac{1}{(2\pi)^2}\int dt dt' e^{-i\omega t}e^{-i\omega' t'}\langle c\delta(t-t')\rangle = \frac{1}{2\pi}c\delta(\omega+\omega') \tag{754}$$

we get

$$\langle v(t)v(t')\rangle = \int d\omega d\omega' \frac{1}{i\omega+\lambda}\frac{1}{i\omega'+\lambda}e^{i\omega t}e^{i\omega' t'}\frac{1}{2\pi}c\delta(\omega+\omega') = \int d\omega \frac{1}{\lambda^2+\omega^2}\frac{c}{2\pi} = \frac{c}{2\lambda}e^{-|t-t'|} \tag{755}$$

and therefore $\langle r^2(t)\rangle$ is linearly growing at large t

$$\frac{d}{dt}\langle r^2(t)\rangle = \frac{d}{dt}\left\langle \int_0^t v(\tau)d\tau \int_0^t v(\tau')d\tau'\right\rangle = 2\int_0^t \frac{c}{2\lambda}e^{-\lambda(t-\tau')}d\tau' = \frac{c}{\lambda^2}(1-e^{-\lambda t}) \tag{756}$$

The constant $c = \frac{C}{M^2} = 2\lambda\langle v^2\rangle$ and at $t\to\infty$ the growth is $\langle r(t)^2\rangle = 2t\frac{\langle v^2\rangle}{\lambda} = t\frac{2\cdot 3k_bT/M}{6\pi\eta R/M} = t\frac{k_bT}{\pi\eta R}$

4.18 s/s00m3 T

a). We will start from partition sum

$$Z = (1+\lambda)^N(1+\lambda e^{\frac{-\epsilon}{T}})^{N_i} \tag{757}$$

(here $\lambda = e^{\frac{\mu}{T}}$) Then

$$F = -T\log Z = -TN[\log(1+\lambda) + \rho\log(1+\lambda e^{\frac{-\epsilon}{T}})] \tag{758}$$

and

$$f = \frac{F}{N} = -T[\log(1+\lambda) + \rho\log(1+\lambda e^{\frac{-\epsilon}{T}})] \tag{759}$$

Now we will determine μ from the condition that total number of particles equal to N:

$$N = N[\frac{\lambda}{1+\lambda} + \rho\frac{\lambda}{e^{\frac{\epsilon}{T}}+\lambda}] \tag{760}$$

and

$$\lambda(\rho,T,\epsilon) = \frac{1-\rho+\sqrt{(1-\rho)^2+4\rho e^{\frac{\epsilon}{T}}}}{2\rho} \tag{761}$$

b). Density of defects is just

$$n(T) = \rho\frac{\lambda}{e^{\frac{\epsilon}{T}}+\lambda} = \frac{\rho}{1+\lambda^{-1}e^{\frac{\epsilon}{T}}} \tag{762}$$

c). When $T\to\infty$ then

$$\lambda(T) \to \rho^{-1}[1+\frac{\epsilon}{T}\frac{\rho}{1+\rho} + O(\frac{\epsilon^2}{T^2})] \tag{763}$$

and

$$n(T) = \frac{\rho}{1+\rho} - \frac{\epsilon}{T}(\frac{\rho}{1+\rho})^2 + O(\frac{\epsilon^2}{T^2}) \tag{764}$$

When $T\to 0$ then

$$\lambda(T) \to \rho^{-1/2}e^{\frac{\epsilon}{2T}} + O(1) \tag{765}$$

and

$$n(T) = \sqrt{\rho} e^{\frac{-\epsilon}{2T}} + O(e^{\frac{-\epsilon}{T}}) \tag{766}$$

Conclusion: $n(T)$ starts from 0 at zero temperature and increases up to $\frac{\rho}{1+\rho}$ at infinite temperature. d). $C = \frac{dE}{dT}$ and $E = n(T)\epsilon$. So

$$C = \epsilon n'(T) = \frac{\epsilon}{\rho T^2} e^{\frac{\epsilon}{T}} \lambda^{-1} n^2(T)[1 - \frac{e^{\frac{\epsilon}{T}}}{\sqrt{(1-\rho)^2 + 4\rho e^{\frac{\epsilon}{T}}}}] \tag{767}$$

Finally

$$S(T) = \rho N[\lambda e^{\frac{\epsilon}{T}} \log(\lambda e^{\frac{\epsilon}{T}}) - \log(1 + \lambda e^{\frac{\epsilon}{T}})] \tag{768}$$

4.19 s/s01j1 T

s01j1 a).$dQ = dU + PdV - \mu dN$ Since $dN = 0$ (container is closed), $U = C_v T, dT = 0$ (where $C_v = \frac{JkT}{2}$, J-numer degrees of freedom) and $PV = NkT$(ideal gas, N-total number of particles in the container) total heat $Q = NkT_1 ln(\frac{V_2}{V_1}) < 0$-the system gives heat. b). For the first container

$$dQ = C_v dT + NkT\frac{dV}{V} \tag{769}$$

For the second

$$dQ' = C_v dT' + NkT'\frac{dV'}{V'}, \qquad dV' = -dV, \quad dQ' = -dQ, \quad V + V' = V_1 + V_2 \tag{770}$$

We want to maximize mechanical work $\int dV(P - P') = \int dV Nk(\frac{T}{V} - \frac{T'}{V_1+V_2-V})$. To do it we can maximize the differential in the last formula treating T as maximal as possible and T' as minimal as possible. In the process we consider at first both containers had the same temperature. Then first container started to expand and it's temperature T decreases, when second decreases it's volume, but increase it's temperature T'. The containers exchange heat dQ attempting to increase T and decrease T' making them equal. But T could not be large then T' since at the moment, when $T = T'$ heat flow stops. Thus system produce maximal amount of mechanical work if the expansion will be slow and containers will be able to flatten their temperatures. Now we treat $T' = T$ and

$$dA = dV(P - P') = NkT(\frac{dV}{V} - \frac{dV'}{V'}) = JNkdT \tag{771}$$

Solving this equation one has

$$T = T_1(\frac{V_2 V}{V_1(V_1 + V_2 - V)})^{\frac{1}{J}} \tag{772}$$

According to previous equation

$$W = JNk(T_1 - T(V)) \tag{773}$$

Obviously maximal work will be done if we stop when the volume of both containers coincides $2V = V_1 + V_2$. Then

$$W = JNkT_1(1 - (\frac{V_2}{V_1})^{\frac{1}{J}}) \tag{774}$$

c). Now we want to compare Q and W. Let us introduce the variable $x = \frac{V_2}{V_1} \in (0..1]$. Note that $Q(x=1) = W(x = 1) = 0$. According to the hint we can consider

$$\frac{dW}{dx} = -\frac{NkT_1}{x^{\frac{J-1}{J}}} \tag{775}$$

and

$$\frac{dQ}{dx} = -\frac{NkT_1}{x} \tag{776}$$

Since $\frac{dQ}{dx} < \frac{dW}{dx}$ for any $x \in (0..1)$ and $Q(1) = W(1)$ then $W < Q$ for any $x \in (0..1)$.

4.20 s/s01j2 T

a). We treat bound state of two biomolecules as a new type of biomolecules. Obviously chemical potential of the molecule of new type is just a sum of chemical potential of the parts, shifted by bound energy. Then the grand sum is

$$Z = Z_A(T,\mu_A) Z_B(T,\mu_B) Z_C(T,\mu_C) Z_{AC}(T,\mu_A+\mu_C+\epsilon_{AC}) Z_{BC}(T,\mu_B+\mu_C+\epsilon_{BC}) \tag{777}$$

We also assume that ϵ_{AC} and ϵ_{BC} are large enough so there are no pure C molecules and we drop Z_C from the sum.

We also believe that classical (Boltzman) distribution is valid here and that the energy levels of biomolecules do not depend on their type: they are very massive and large-thus only their position could contribute. (Generally we could also assume that partition function of boundstate is not equal to partition function of a part, but their product. This case describes (almost) independent structures.)

Now (all partition functions are the Boltzman summ of the form $Z = \sum_E e^{-E/T}$)

$$N_A = Z_A(T) e^{\frac{\mu_A}{T}} + Z_A(T) Z_C e^{\frac{\mu_A+\mu_C-\epsilon_{AC}}{T}} \tag{778}$$

$$N_B = Z_B(T) e^{\frac{\mu_B}{T}} + Z_B(T) Z_C e^{\frac{\mu_B+\mu_C-\epsilon_{BC}}{T}} \tag{779}$$

and

$$N_C = Z_A Z_C(T) e^{\frac{\mu_A+\mu_C-\epsilon_{AC}}{T}} + Z_B(T) Z_C e^{\frac{\mu_B+\mu_C-\epsilon_{BC}}{T}} \tag{780}$$

Generally each Z could be multiplied by Z_0 (for center of mass) but we absorb it to unknown N_i-number of particles.

We absorb corresponding Z to μ and have

$$f_A = \frac{1}{1+e^{\mu_B-\mu_A+\epsilon_{BC}-\epsilon_{AC}}} \tag{781}$$

$$f_B = \frac{1}{1+e^{\mu_A-\mu_B+\epsilon_{AC}-\epsilon_{BC}}} \tag{782}$$

b). We already used first assumption about $f_A = 1$ when $\mu_B \to -\infty$. For n_c we have

$$n_C = \frac{n_A}{1+e^{-\frac{\mu_C+\epsilon_{AC}}{T}}} + \frac{n_B}{1+e^{-\frac{\mu_C+\epsilon_{BC}}{T}}} \tag{783}$$

Since we know that n_C should much smaller than n_A and n_B we conclude that we could drop 1 in the denominators and get for f

$$f_A = \frac{1}{1+\frac{n_B}{n_A} e^{\frac{\epsilon_{BC}-\epsilon_{AC}}{T}}} \tag{784}$$

c). Let $\frac{\epsilon_{BC}-\epsilon_{AC}}{kT} = x$. Then

$$0.1 = \frac{1}{1+0.01e^x} \tag{785}$$

or $x = \log 900 \sim 6.8$. Using that $10000K \sim 1eV$ we have $\epsilon_{BC}-\epsilon_{AC} \sim 0.2$eV.

4.21 s/s01j3 T

s01j3 a). Hamiltonian has a simmetry $S_i \to -S_i$ thus $< S_i >= 0$. Effective hamiltonian is

$$H = \sum_i^N (S_{2i-1} S_{2i} lncsh(S_{2i-1}+S_{2i})) + Nln2 \tag{786}$$

b).Hamiltonian of single triangle is

$$H = \frac{1}{2}(S_1+S_2+S_{1-2})^2 - \frac{3}{2} \tag{787}$$

Thus we have $6 = 3 \times 2$ ground states, where 3 is a number of possible choices of particle with spin opposite to other two particles and 2 is a number of possible spins of this particle. c). Let V_N be the number of ground states for the system of N triangles with given value of spin in the left bottom corner (obviously V_N does not depend on this value). According to b. first (at the left) triangle has 3 configuration of spins with minimal energy. Thus

$$V_N = 3V_{N-1} \tag{788}$$

And $V_0 = 1$. Now we simply multiply this result by 2 as the number of different values of the spin in the left bottom corner. Answer:

$$2 \times 3^N \tag{789}$$

d). We consider $< S_i S_j >$. Let i be a right corner of p-th triangle and j be the left corner of q-th triangle. Then there are $k = N - p - q$ full triangles between i and j $(i < j)$. Now we want to calculate $W_k(s)$-number of ground states in the system of k triangles with given boundary conditions: $s = 1$ if spins on the boundary coincides and -1 otherwise. We already know that $W_k(1) + W_k(-1) = 2 \times 3^k$.

From the previous speculations (slightly generalize them) we can conclude that

$$W_{k-1}(1) = W_k(1) + 2W_k(-1) \tag{790}$$

and similarly

$$W_{k-1}(-1) = W_k(-1) + 2W_k(+1) \tag{791}$$

Or the problem is equivalent of the problem of calculating

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^k \tag{792}$$

The eigenvalues of this matrix is -1 and 3 with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{793}$$

and

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{794}$$

respectively. As a result $W_k(+) = [3^k + (-1)^k]$ and $W_k(-) = [3^k + (-1)^k]$. Finally

$$< S_i S_j > = \frac{V_p(W_k(+) - W_k(-))V_q}{2V_N} = \frac{(-1)^k}{3^k} \tag{795}$$

For $k = 0$ we get 1 as expected.

4.22 s/s01m1 V

a) Consider the black body radiation. From the Bose-Einstein distribution:

$$d\,\varepsilon = \frac{c}{4} 2 \frac{d^3p}{h^3} \frac{\varepsilon}{e^{\frac{h\nu}{kT}} - 1} dS = \frac{2\pi\nu^2}{c^2} \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} dS d\nu \tag{796}$$

From the extremum ν_{max} in the power spectra follows the relation

$$h\nu_{max} = \alpha kT, \tag{797}$$

where c_1 is the dimensionless numerical constant (obtained from solving the extremum equation).

At the low frequencies the spectrum has asymptotic

$$dE = \frac{2\pi\nu^2}{c^2} kT d\nu dS \tag{798}$$

The energy emitted from the area of the size of one wavelength, during the the time interval of the one period is

$$dE = 2\pi kT \frac{d\nu}{\nu}. \tag{799}$$

From this measurement one can find the Boltzman constant

$$k = \frac{1}{2\pi T}\nu \frac{dE}{d\nu} \tag{800}$$

and the Avogadro number $A = \frac{R}{k}$. After k has been found, from (797) follows the formula for h:

$$h = \frac{\alpha kT}{\nu_{max}} \tag{801}$$

b) Since A is dimensionless, then whatever they mean by "pure thermodynamics mean" one needs something that has dimension of time to get the dimension of $h = energy \times time$ from the dimension of Q (energy). By heating a box with photons from zero temperature and measuring the heat input one gets the relation

$$h = \left(\frac{V}{Q}\frac{8\pi^5 (kT)^4}{15c^3}\right)^3. \tag{802}$$

The factor V/c^3 can be expressed as τ^3 where τ is a time for light to cross the box.

4.23 s/s01m2 V

The canonical partition function at constant pressure (we use notation $P = -t$, where t is tension, and V for the length)

$$1 = \sum e^{\frac{F-E_n}{T}} = \sum e^{\frac{\Phi - pV - E_n}{T}} \tag{803}$$

from which follows

$$\Phi = -TN\log\left(e^{-\frac{E_a+Pl_a}{T}} + e^{-\frac{E_b+Pl_b}{T}}\right) \tag{804}$$

and since $d\Phi = -SdT + VdP$

$$V = \left(\frac{\partial \Phi}{\partial P}\right)_T = N\frac{l_a e^{-\frac{E_a+Pl_a}{T}} + l_b e^{-\frac{E_b+Pl_b}{T}}}{e^{-\frac{E_a+Pl_a}{T}} + e^{-\frac{E_b+Pl_b}{T}}} \tag{805}$$

4.24 s/s01m3 V

a) In the magnetic field the particle with hamiltonian

$$H = \frac{1}{2m}\left(p_x - \frac{e}{c}By\right)^2 + \frac{1}{2m}p_y^2 \tag{806}$$

has energy levels of the harmonic oscillator $E_k = \hbar\omega(n+\frac{1}{2})$ with $\omega = \frac{eB}{mc}$. The coordinate of the center of the circle is $y = \frac{cp_x}{eB}$, the motion is confined into $0 < y < L_y$ if $0 < p_x < L_y\frac{eB}{c}$. The number of such states $g = \frac{L_x}{2\pi\hbar}L_y\frac{eB}{c}$.

b) From the canonical grand distribution

$$\Omega = -T\sum \log(1 + e^{\frac{\mu - E_k}{T}}) \tag{807}$$

in the low density regime we have

$$\Omega = -T\sum e^{\frac{\mu - E_k}{T}} = -T\frac{V}{(2\pi\hbar)^2}\frac{eB}{c}(2\pi mT)^{\frac{1}{2}}e^{\beta\mu}\frac{1}{\sinh\frac{\hbar\omega}{2T}} = -PV \tag{808}$$

thus

$$P = T\frac{1}{(2\pi\hbar)^2}\frac{eB}{c}(2\pi mT)^{\frac{1}{2}}e^{\beta\mu}\frac{1}{\sinh\frac{\hbar\omega}{2T}} \tag{809}$$

from $N = -\frac{\partial\Omega}{d\mu}$ one gets that at $B = 0$, $\Omega = -NT$ and from $\chi = -\frac{\partial^2\Omega}{\partial B^2}$ one finds

$$\chi = -\frac{1}{6}\frac{N}{T}\left(\frac{\hbar e}{mc}\right)^2 \tag{810}$$

The gas is diamagnetic (Landau).

4.25 s/s02j1 T

a). Using that

$$P = -\frac{\partial F}{\partial V} = \frac{RT}{V-b} - \frac{a}{V^2} \tag{811}$$

we have

$$F = -RT\log(V-b) - \frac{a}{V} + f(T) \tag{812}$$

Function $f(T)$ is unknown, but using that

$$C_v = \frac{\partial U}{\partial T} \tag{813}$$

and

$$U = -T^2\frac{\partial}{\partial T}\frac{F}{T} \tag{814}$$

we get

$$C_v = -T\frac{\partial^2 F}{\partial T^2} = -Tf'' \tag{815}$$

or

$$f(T) = -C_v T\log T + cT \tag{816}$$

where c is unknown constant. Eventually

$$F = -RT\log(V-b) - \frac{a}{V} - C_v T\log T + cT \tag{817}$$

b). Using the low of conservation of energy applied for small amount of gas traveled from one reservoir to another:

$$dE = dU_1 - dU_2 = dA_1 - dA_2 = -P_1 dV_1 + P_2 dV_2 \tag{818}$$

Using that $dP_i = 0$ we get $dH_1 = dH_2$ c).

$$H = C_v T + \frac{RTV}{V-b} - \frac{2a}{V} \tag{819}$$

d). From b. we know that $H = const$ and this explains that

$$\Delta T = \Delta V\left(\frac{\partial T}{\partial V}\right)_H \tag{820}$$

So T_{inv} is the temperature when $\left(\frac{\partial T}{\partial V}\right)_H = 0$ Using that

$$dH = 0 = dT(C_v + \frac{RT}{V-b}) + dV(\frac{2a}{V} - \frac{RT}{(V-b)^2}) \tag{821}$$

we have

$$\left(\frac{\partial T}{\partial V}\right)_H = \frac{Rb}{(V-b)^2}\frac{T-T_{int}}{C_v + RV/(V-b)} \tag{822}$$

where $T_{int} = \frac{2a}{bR}(1-\frac{b}{V})^2$

At the last step we will determine ΔV from ΔP. Using that

$$H(P,V) = \frac{C_v}{R}(V-b)(P+\frac{a}{V^2}) - \frac{a}{V} + PV \tag{823}$$

we obtain

$$\Delta V = -\Delta P \frac{[V + \frac{C_v}{R}(v-b)]}{[\frac{C_v}{R}(P-\frac{a}{V^2}) + (P + \frac{a}{V^2} + \frac{2abC_v}{RV^3})]} \tag{824}$$

For relatively large pressure ($PV^2 > a$) volume increases and temperature increases/decreases depending on $T > or < T_{int}$.

4.26 s/s02j2 T

s02j2 a).

$$Z = \int d\nu(E) ln(1 + 2ch(\frac{\mu_B H}{T})e^{\frac{\mu - E}{T}} + e^{\frac{2(\mu-E)}{T}}) \tag{825}$$

$$N_{\pm} = \int d\nu(E) \frac{e^{\frac{\mu - E \pm \mu_B H}{T}} + e^{\frac{2(\mu-E)\mu_B H}{T}}}{1 + 2ch(\frac{\mu_B H}{T})e^{\frac{\mu-E}{T}} + e^{\frac{2(\mu-E)}{T}}} \tag{826}$$

b). When $T \to 0$ $\mu \to \epsilon_F$ by definition of $\mu(T=0)$. Then

$$N_+ = \frac{4\pi V}{3(2\pi\hbar)^3}(2m(\epsilon_F + \mu_B H))^{3/2} \tag{827}$$

and

$$N_- = \frac{4\pi V}{3(2\pi\hbar)^3}(2m\epsilon_F - \mu_B H)^{3/2} \tag{828}$$

c).

$$N = N_+ + N_- \tag{829}$$

and

$$M = (N_+ - N_-)\mu_B \tag{830}$$

d).

$$\chi = \frac{3N\mu_B^2}{2\epsilon_F} \tag{831}$$

4.27 s/s02j3 T

s01j1 a). $E = \hbar w M$

$S = klnW(M)$

b). System is isolated, in the Stirling approximation

$$n! \approx \sqrt{2\pi n} n^n e^{-n} \tag{832}$$

$$S = klnW(M) = k[(N+M-1/2)ln(N+M-1) - (M+1/2)lnM + Const] \tag{833}$$

Then, by definition

$$\frac{1}{T} = \frac{dS}{dE} = \frac{k}{\hbar w}[ln(\frac{M+N-1}{M}) + \frac{1}{2(M+N-1)} - \frac{1}{2M}] \tag{834}$$

We also assume that $N, M >> 1$. Then

$$\frac{M}{N} = \frac{1}{e^{\frac{\hbar w}{kT}} - 1} \tag{835}$$

as expected. To get the same result through Boltzman partitions it is convenient to notice that

$$\sum_{n_1=0}^{\infty} ... \sum_{n_N=0}^{\infty} \delta(M - \sum_{i=1}^{N} n_i) = \frac{(N+M-1)!}{M!} \tag{836}$$

In this way our partition sum is

$$Z = \sum_M e^{\frac{-\hbar w M}{kT}} W(M) = \frac{1}{(N-1)!(e^{\frac{\hbar w}{kT}} - 1)^N} \tag{837}$$

Then

$$\frac{<M>}{N} = \frac{<E>}{\hbar w N} = \frac{d \log Z}{\hbar w N dT^{-1}} = \frac{1}{e^{\frac{\hbar w}{kT}} - 1} \tag{838}$$

c). Straightforwardly

$$C_v = \frac{dE}{dT} = \frac{\hbar^2 w^2 N}{kT^2} \frac{e^{\frac{\hbar w}{kT}}}{(e^{\frac{\hbar w}{kT}} - 1)^2} \tag{839}$$

d). This was already derived in b). Now we give alternative derivation using Boltzman partition. At first we note that instead of coordinates E, M where $S = lnW(M)$ in coordinates T, M entropy is $S = \sum PlnP$ or

$$S = -k \sum_{M=0}^{\infty} \frac{W(M) e^{\frac{-\hbar w M}{kT}}}{Z} ln[\frac{e^{\frac{-\hbar w}{T}}}{Z}] \tag{840}$$

where

$$Z = \sum_{M=0}^{\infty} e^{\frac{-\hbar w M}{kT}} W(M) \tag{841}$$

This sum could be sum up:

$$S = k(lnZ + \sum_{M=0}^{\infty} \frac{W(M)\hbar w M e^{\frac{-\hbar w M}{kT}}}{kTZ}) = k\frac{d(TlnZ)}{dT} \tag{842}$$

We want to check whether

$$\frac{dS}{dE} = \frac{dS}{dT}\frac{dT}{dE} = k\frac{d^2(TlnZ)}{dT^2}\frac{dT}{dE} = \frac{1}{T} \tag{843}$$

Or in another words

$$\frac{d^2(TlnZ)}{dT^2} = \frac{dE}{kTdT} = \frac{d}{TdT}\frac{T^2 dlnZ}{dT} \tag{844}$$

according to the definition of $E = k\frac{T^2 dlnZ}{dT}$. But

$$\frac{d^2(TlnZ)}{dT^2} = \frac{d}{TdT}\frac{T^2 dlnZ}{dT} = 2\frac{dlnZ}{dT} + \frac{d^2 lnZ}{dT^2} \tag{845}$$

is just the trivial identity!

e). Any particular configuration could be imagined as the row of objects: bosons and oscillators. Then all bosons to the left of any particular oscillator (up to the next oscillator) belong to it. Thus no bosons should be to the right of the "rightest" oscillator. We assume that number of the "rightest" oscillator is N -the number is not important here because according to our assumption all oscillators are equal to each other. Then number of series of M bosons and $N-1$ oscillators (the last is already on the right side) is just $(N+M-1)!$. But here all objects are equal. We have to dived this number by number of permutations of bosons itself $M!$ (without changing the series) and also oscillators $(N-1)!$. Result is

$$W(M,N) = \frac{(M+N-1)!}{M!(N-1)!} \tag{846}$$

4.28 s/s02m1 T

s01m3 a).

$$\rho(h) = \rho_0 e^{\frac{-mgh}{T}} \tag{847}$$

Derivation: we assume that gas is classical (governed by Boltzman statistics). Then using that μ doesn't depend on h and that "effective" $\mu(h) = \mu - mgh$ we derive the dependence $\frac{N(h)}{V}$ assuming that we consider very small volume where we can neglect the dependence of potential energy mgh on height h. b). This is usual Maxwell distribution (it doesn't depends on h up to normalization)

$$F(p) = e^{\frac{-E(p)}{T}} e^{\frac{\mu(h)}{T}} \tag{848}$$

c). Under **strictly** ideal (not classic) gas we have to understand gas governed by bose (fermi) statistics. Then effective $\mu(h)$ is still the same, but dependence $\rho(h)$ is more complicated. It is the function, which expressed total number of particles through chemical potential

$$N = \frac{dlnZ}{d\mu} \tag{849}$$

with substituted effective potential $\mu(h) = \mu - mgh$. Nevertheless this function is not elementary for both B and F statistics.

$F(p)$ remains the same (up to normalization). It is

$$F(p,h) = \frac{e^{\frac{-E(p)+\mu(h)}{T}}}{Z(T,\mu(h))} \sim e^{\frac{-E(p)}{T}} \tag{850}$$

d). The same up to normalization constant which is the ration of densities at bottom and at the top. How to calculate the densities is explained in c.

4.29 s/s02m2 T

s02j2 a).Using approximation of Boltzman gas: total number of protons is ($E_0 = 13.6$ eV)

$$\frac{N_p}{V} = \left(\frac{M_pT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_p}{T}} + \left(\frac{M_HT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_p+\mu_e+E_0}{T}} \tag{851}$$

and electrons is

$$\frac{N_p}{V} = \left(\frac{M_eT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_e}{T}} + \left(\frac{M_HT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_p+\mu_e+E_0}{T}} \tag{852}$$

From this moment we will treat $M_e << M_p$ and thus $M_H = M_p$. We know that at the temperature $T_0 = 0.3$eV number of atoms is equal to the number of free protons. Or

$$\left(\frac{M_HT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_p}{T}} = \left(\frac{M_pT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_p+\mu_e+E_0}{T}} \tag{853}$$

Thus $\mu_e(T_0) = -E_0$. To determine both quantities $\frac{N_e}{V}$ and $\frac{N_p}{V}$ we need once more equation.

How to get it?

Let us remind that our Universe is electrically neutral. And that is why the next equation is

$$\frac{N_e}{V} = \frac{N_p}{V} \tag{854}$$

This is not a result of statistical physics, but new assumption about system we consider. Using this one could simply gets

$$\frac{N_e}{V} = \frac{N_p}{V} = 2\left(\frac{M_e T}{2\pi\hbar^2}\right)^{3/2} e^{\frac{-E_0}{T}} \tag{855}$$

We learn that our approximation is correct since the density we deal with is $e^{\frac{-E_0}{T}} \sim E^{-45}$ times (!) than critical one. Density of free electrons is equal to density of free protons and to the density of Hydrogen atoms and is $\frac{N_e}{2V}$ b). From Plank formula density of photons is

$$\frac{N_{ph}}{V} = \frac{T^3}{2\pi^2 c^3 \hbar^3} \int_0^\infty \frac{dx x^2}{e^x - 1} \tag{856}$$

Let us compare N_e and N_{ph}. Roughly speaking

$$\frac{N_e}{N_{ph}} = \frac{m_e}{T}^{3/2} e^{-E_0/T} \tag{857}$$

At $T = T_0 = 0.3\text{eV}$ we have

$$\frac{N_e}{N_{ph}} \sim \frac{511}{0.3}^{3/2} e^{-45} \sim \times 10^4 \times 10^{-20} << 17 \tag{858}$$

Or $N_e << N_{ph}$.

4.30 s/s02m3 T

s02m3

a). $S = \log g$

where g is a number of microstates with given energy. At the zero temperature energy is also zero. If $S \neq 0$ then there are more than one ground states of the ice's hamiltonian. This is of course possible, but unusual.

b).There are $2N$ bonds (this could be get by simple combinatorics, but also from main formula of chemistry H_2O) and each bond has exactly 2 quantum states. Result $g = 2^{2N}$ and $S = 2N \log 2$

c). Let us consider one particular oxygen atom (and bonds end on it) and forget about all others (according the assumption). There are 2^4 states of such system if we want to take into account all other bonds as well). The number of configurations, when exactly two of hydrogen atom are close to this oxygen atom are $6 = \frac{4!}{2!}$ (bonds are different, no matter in what oder we will select them). Fraction is

$$\frac{6}{2^4} = \frac{3}{8} \tag{859}$$

Now

$$g = \frac{N}{2} \log 6 \tag{860}$$

My result is twice larger than Pauling's!!!!!

4.31 s/s03m1 T

$$TdS = dQ = dU + dA \tag{861}$$

where $dA = -fdx$. When x is fixed $dA = 0$ and

$$TdS = dQ = C(x)dT = dU = \frac{\partial U}{\partial T} dT \tag{862}$$

or

$$C(x) = \frac{\partial U}{\partial T} \tag{863}$$

and thus

$$U = \frac{A(x)}{2}T^2 + g(x) \tag{864}$$

where $g(x)$ unknown arbitrary function.

At the constant zero temperature

$$TdS = dQ = 0 = dU + f(x, T = 0)dx = [g' - f(x, 0)]dx \tag{865}$$

thus

$$g = \frac{\mu}{2}x^2 + Const \tag{866}$$

Now

$$U = \frac{A(x)}{2}T^2 + \frac{\mu}{2}x^2 \tag{867}$$

After we have determined exact form of U (up to additive constant) we can calculate S:

$$TdS = dU - fdx = dT[A(x)T] - dx[-\frac{A'(x)}{2}T^2 - \alpha T + \beta xT] \tag{868}$$

Let us divide both sides of these equation by T. Now for the right side to be full differential following property of S should be satisfied

$$\frac{\partial^2 S}{\partial T \partial x} = \frac{\partial^2 S}{\partial x \partial T} \tag{869}$$

or

$$A'(x) = \frac{A'(x)}{2} \tag{870}$$

or

b). $A' = 0$ and $A = const$!

a).$\frac{\partial S}{\partial x} = \alpha - \beta x$

c).$\frac{\partial S}{\partial T} = A$

$$S = AT + \alpha x - \frac{\beta}{2}x^2 + B \tag{871}$$

d). Zero tension $f = 0 = \mu x - \alpha T + \beta xT \Rightarrow dx[\mu + \beta T] = dT[\alpha - \beta x]$

$$C_F = T\left(\frac{\partial S}{\partial T}\right)_{f=0} = T[\frac{\partial S}{\partial T} + \frac{\partial S}{\partial x}\frac{\alpha - \beta x}{\mu + \beta T}] = \tag{872}$$

$$C_F = T[A + (\alpha - \beta x)\frac{\alpha - \beta x}{\mu + \beta T}] \tag{873}$$

Using that $f = 0$ implies $x = \frac{\alpha T}{\mu + \beta T}$ then

$$C_F] = T[A + \frac{\alpha^2\mu^2}{(\mu + \beta T)^3}] \tag{874}$$

4.32 s/s03m2 T

a). Since we assume that density is uniform we substitute the spherical star by a cube of size L with periodic boundary conditions. Than $k = \frac{2\pi n_e}{L}$ and total number of electrons N should be equal to $\frac{4\pi}{3} n_e^3$, where

$$E_f = \left(\frac{2\pi n_e}{L}\right)^2 \frac{1}{2m} = \frac{1}{m}\left(\frac{3\pi^2 N}{V\sqrt{2}}\right)^2 \tag{875}$$

And total kinetic energy is

$$E = \int_0^{n_e} dnn^2 \frac{k^2}{2m} = \frac{3(2\pi\hbar)^2 N}{10m}\left(\frac{3N}{4\pi}\right)^{2/3} \tag{876}$$

b). Solving the equation

$$\partial_R(U_k + U_g) = 0 \tag{877}$$

we simply find

$$R = \frac{(2\pi\hbar)^2}{2^{5/2} m_p^{5/2} m_e G}\left(\frac{3}{4\pi}\right)^{4/3} M^{-1/3} \sim M^{-1/3} \tag{878}$$

c). n_e remains the same. E_f now is equal to

$$E_f = ck_e = c\frac{2\pi n_e}{L} = 2\pi c\left(\frac{3N}{4\pi V}\right)^{1/3} \tag{879}$$

and total energy is

$$E = 4\pi\int_0^{n_e} dnn^2 ck = \frac{3(2\pi\hbar)cN}{4}\left(\frac{3N}{4\pi V}\right)^{1/3} \tag{880}$$

d). This condition is $U_k < U_g$ (this condition doesn't depend on R). It yields

$$M^{\frac{2}{2}} > \frac{5(2\pi\hbar c)}{8 2^{1/3} m^{5/3} G}\left(\frac{3}{4\pi}\right)^{2/3} \tag{881}$$

4.33 s/s03m3 T

a). Sum for classical particle is

$$Z_{cl} = \sum_n e^{\frac{-E}{T}} \tag{882}$$

Here n is a set of quantum numbers n_1, n_2, n_3 which specify quantum state of particle by

$$k_i = \frac{2\pi n_i \hbar}{L} \tag{883}$$

and $V = L^3$. Then partition sum

$$Z_{cl} = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3=-\infty}^{\infty} e^{\frac{-E}{T}} \tag{884}$$

could be substituted by integral over d^n. Really using variable k instead of n we will get at first sum over lattice with scale $\frac{2\pi\hbar}{L}$ and if m and T finite sum goes to integral as scale of lattice approaches zero.

Taking this gaussian integral now we will have

$$Z_{cl} = \int d^3 n e^{\frac{-E_n}{T}} = V/\lambda^3 \tag{885}$$

as expected. b).

Going back to the sum we have

$$Z = \frac{V}{\lambda^3} \sum e^{-x^2} \tag{886}$$

where we sum over 3D lattice with size (roughly) $\frac{\hbar}{L\sqrt{mT}}$. The sum could not be substituted by integral if the function we want to integrate changes sufficiently at lattice scale. Exponent change sufficiently if x changes by 1 and our approximation breaks if $\frac{\hbar}{L\sqrt{mT}} \approx 1$ or large. Result

$$T <\sim \frac{\hbar^2}{mV^{2/3}} \tag{887}$$

c). Two particle partition sum for quantum particles differs from the same sum for classical particles because in the quantum case particles are identical.

Thus for bosons

$$Z_B = \frac{1}{2} \sum_{n \neq m} e^{\frac{-E_n}{T}} e^{\frac{-E_m}{T}} + \sum_n e^{\frac{-2E_n}{T}} \tag{888}$$

for fermions

$$Z_F = \frac{1}{2} \sum_{n \neq m} e^{\frac{-E_n}{T}} e^{\frac{-E_m}{T}} \tag{889}$$

Using that Thus for bosons

$$2E_n(m) = E_n(\frac{m}{2}) \tag{890}$$

we have

for bosons

$$Z_B = \frac{1}{2} \sum_{nm} e^{\frac{-E_n}{T}} e^{\frac{-E_m}{T}} + \frac{1}{2} \sum_n e^{\frac{-2E_n}{T}} = \frac{1}{2} Z(m,T)^2 + \frac{1}{2} Z(\frac{m}{2}, T) \tag{891}$$

for fermions

$$Z_F = \frac{1}{2} \sum_{nm} e^{\frac{-E_n}{T}} e^{\frac{-E_m}{T}} - \frac{1}{2} \sum_n e^{\frac{-2E_n}{T}} = \frac{1}{2} Z(m,T)^2 - \frac{1}{2} Z(\frac{m}{2}, T) \tag{892}$$

d). Using explicit form of Z

$$Z(T,V,m) = \frac{V}{\lambda^3} \tag{893}$$

$$E = T^2 \frac{d \log Z}{dT} = \frac{3}{2} T [1 + \frac{1}{1 \pm Z^{-1}(m,2T)}] \tag{894}$$

$$C = \frac{dE}{dT} = \frac{3}{2} [1 + \frac{1}{1 \pm Z^{-1}(m,2T)}] \pm \frac{9Z^{-1}(m,2T)}{4(1 \pm Z^{-1}(m,2T))^2} \tag{895}$$

As was expected (thanks to Daniel) at the classical limit $Z \to \infty$ the pure classical result restores.