2007 Part I Q6b Comp.

$$a \quad \frac{U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^{n}}{\delta t} + a\left(\frac{U_{j+1}^{n+\frac{1}{2}} - U_{j}^{n+\frac{1}{2}}}{\delta x}\right) = 0$$

$$U_{j+\frac{1}{2}}^{n} \equiv \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) \qquad U_j^{n+\frac{1}{2}} \equiv \frac{1}{2}\left(U_j^n + U_j^{n+1}\right)$$

$$\Rightarrow \frac{U_j^{n+1} + U_{j+1}^{n+1} - U_j^n - U_{j+1}^n}{2\delta t} + \frac{a}{2\delta x}\left(U_{j+1}^n + U_{j+1}^{n+1} - U_j^n - U_j^{n+1}\right) = 0$$

b. $U_j^n \longrightarrow r^n \tilde{u}_k e^{ikx_j}$ $\quad x_j = j\,\delta x \quad k\,\delta x \equiv \theta \Rightarrow U_j^n \to r^n \tilde{u}_k e^{ij\theta}$

$$r + re^{i\theta} - 1 - e^{i\theta} = \frac{a\delta t}{\delta x}\left(e^{i\theta} + re^{i\theta} - 1 - r\right) \qquad \text{let } s \equiv \frac{a\delta t}{\delta x}$$

$$r(1+e^{i\theta}) - (1+e^{i\theta}) = rs(e^{i\theta}-1) + s(e^{i\theta}-1)$$

$$r\left(1+e^{i\theta} - s(e^{i\theta}-1)\right) = 1+e^{i\theta} + s(e^{i\theta}-1)$$

$$r = \frac{1 + e^{i\theta} + s(e^{i\theta}-1)}{1+e^{i\theta} - s(e^{i\theta}-1)} \qquad \text{divide by } e^{i\theta/2}$$

$$r = \frac{\cos\frac{\theta}{2} + is\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - is\sin\frac{\theta}{2}}$$

For stability, $|r|^2 < 1$

$$|r|^2 = rr^* = \frac{\cos\frac{\theta}{2} + is\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - is\sin\frac{\theta}{2}} \times \frac{\cos\frac{\theta}{2} - is\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + is\sin\frac{\theta}{2}} = 1$$

$$|r|^2 = 1$$

Always stable!

c. Truncation error:

$$U_j^{n+1} = U_{j+\frac{1}{2}}^{n+\frac{1}{2}} + (\delta t/2)\frac{\partial U}{\partial t}\bigg|_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\delta x/2)\frac{\partial U}{\partial x} + \frac{(\delta t/2)^2}{2!}\frac{\partial^2 U}{\partial t^2} - \frac{2(\delta t/2)(\delta x/2)}{2!}\frac{\partial^2 u}{\partial x\partial t} + \frac{(\delta x/2)^2}{2!}\frac{\partial^2 U}{\partial x^2} + \cdots \; O(\delta^3)$$

$$U_{j+1}^{n+1} = '' + '' + '' + '' + '' + '' + \cdots$$

$$U_j^n = '' - '' - '' + '' + '' + '' + \cdots$$

$$U_{j+1}^n = '' - '' + '' + '' - '' + '' + \cdots$$

Third order terms:

$$U_j^{n+1} = \dots + \frac{(\delta t/2)^3}{3!}\frac{\partial^3 U}{\partial t^3} \quad \frac{3(\delta t/2)^2(\delta x/2)}{3!}\frac{\partial^3 U}{\partial t^2 \partial x} + \frac{3(\delta t/2)(\delta x/2)^2}{3!}\frac{\partial^3 U}{\partial t \partial x^2} - \frac{(\delta x/2)^3}{3!}\frac{\partial^3 U}{\partial x^3} + O(\delta^4)$$

$$U_{j+1}^{n+1} = \dots + \quad '' \quad + \quad '' \quad + \quad '' \quad + \quad ''$$

$$U_j^{n} = \dots - \quad '' \quad - \quad '' \quad - \quad '' \quad - \quad ''$$

$$U_{j+1}^{n} = \dots - \quad '' \quad + \quad '' \quad - \quad '' \quad + \quad ''$$

$$U_j^{n+1} + U_{j+1}^{n+1} - U_j^n - U_{j+1}^n = 2\delta t \frac{\partial U}{\partial t} + \frac{1}{12}\delta t^3 \frac{\partial^3 U}{\partial t^3} + \frac{1}{4}\delta t\, \delta x^2 \frac{\partial^3 U}{\partial t \partial x^2} + O(\delta^4)$$

$$U_{j+1}^n + U_{j+1}^{n+1} - U_j^n - U_j^{n+1} = 2\delta x \frac{\partial U}{\partial x} + \frac{1}{12}\delta x^3 \frac{\partial^3 U}{\partial x^3} + \frac{1}{4}\delta t^2 \delta x \frac{\partial^3 U}{\partial t^2 \partial x} + O(\delta^4)$$

$$\Rightarrow \frac{\partial U}{\partial t} + \frac{1}{24}\delta t^2 \frac{\partial^3 U}{\partial t^3} + \frac{1}{8}\delta x^2 \frac{\partial^3 U}{\partial t \partial x^2} + a\frac{\partial U}{\partial x} + \frac{a}{24}\delta x^2 \frac{\partial^3 U}{\partial x^3} + \frac{a}{8}\delta t^2 \frac{\partial^3 U}{\partial t^2 \partial x} = 0$$

Truncation Error: $\delta t^2\left(\frac{1}{24}\frac{\partial^3 U}{\partial t^3}\Big|_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{a}{8}\frac{\partial^3 U}{\partial t^2 \partial x}\right) + \delta x^2\left(\frac{a}{24}\frac{\partial^3 U}{\partial x^3} + \frac{1}{8}\frac{\partial^3 U}{\partial t \partial x^2}\right)$

d. $U_j^{n+1} + U_{j+1}^{n+1} - U_j^n - U_{j+1}^n + s\left(U_{j+1}^n + U_{j+1}^{n+1} - U_j^n - U_j^{n+1}\right) = 0$

$\rightarrow$ Implicit. $\quad (1-s)U_j^{n+1} + (1+s)U_{j+1}^{n+1} = (1+s)U_j^n + (1-s)U_{j+1}^n \equiv r_j$

Given boundary conditions $U_0 = U(0)$, $U_N = u(N)$, this is a simple matrix equation:

$$\begin{bmatrix} 1 & 0 & & & & \\ 0 & 1-s & 1+s & & & \\ & 0 & 1-s & \ddots & & \\ & & 0 & \ddots & 1-s & 1+s \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} U_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ U_N \end{bmatrix}^{n+1} = \begin{bmatrix} u(0) \\ r_1 \\ r_2 \\ \vdots \\ r_{N-1} \\ u(N) \end{bmatrix}$$

- Simpler than tridiagonal; can immediately be back-solved

Assuming the RHS r_j is already computed, it takes $3N$ operations.

But it takes $2N$ multiplications to compute the RHS from the previous timestep

$\Rightarrow$ $5N$ operations required to advance 1 step in time

e. Second order accurate and arbitrarily long timesteps; efficient computation