2003 Part I Q2 Nonneutrals

a. $n(r) = \begin{cases} \hat{n} & 0 \le r < r_p \\ 0 & r > r_p \end{cases}$ $P_\perp(r) = \begin{cases} \hat{n}\hat{T}_\perp (1 - r^2/r_p^2) & 0 \le r < r_p \\ 0 & r > r_p \end{cases}$

Gauss's Law: $\nabla\cdot\vec{E} = 4\pi\rho = -4\pi e\hat{n}$ $r < r_p$

$$E\cdot 2\pi r L = -4\pi e\hat{n}\cdot \pi r^2 L$$

$$E_r = -2\pi r e\hat{n}$$

b. $V_\theta(r) = \omega_r r$ $\omega_r = \text{const}$

$$mn\left(\frac{\partial\vec{V}}{dt}\right) = -en\vec{E} - en\frac{\vec{V}\times\vec{B}}{c} - \nabla_\perp P_\perp$$

$$\frac{d\vec{V}}{dt} = \cancel{\frac{\partial\vec{V}}{\partial t}} + \vec{V}\cdot\nabla\vec{V} = V_\theta\hat{\theta}\cdot\nabla V_\theta\hat{\theta} = V_\theta^2\,\hat{\theta}\cdot\nabla\hat{\theta} = -\frac{V_\theta^2}{r}\hat{r}$$

Radial force balance:

$$-m\hat{n}\omega_r^2 r = +e^2\hat{n}^2\cdot 2\pi r - \frac{e\hat{n}\omega_r r B_0}{c} + \frac{2\hat{n}\hat{T}_\perp r}{r_p^2}$$

$$-\omega_r^2 = +\frac{2\pi\hat{n}e^2}{m} - \omega_r\frac{eB_0}{mc} + \frac{2\hat{T}_\perp}{m r_p^2}$$

$$\Rightarrow -\tfrac{1}{2}\omega_p^2 + \omega_r\omega_c - \omega_r^2 = \frac{2\hat{T}_\perp}{m r_p^2} \qquad \omega_c \equiv \frac{eB_0}{mc}$$

c. $\omega_r^2 - \omega_r\omega_c + \left(\frac{2\hat{T}_\perp}{m r_p^2} + \frac{1}{2}\omega_p^2\right) = 0$

$$\omega_r^\pm = \frac{\omega_c \pm \sqrt{\omega_c^2 - 4\left(\frac{1}{2}\omega_p^2 + \frac{2\hat{T}_\perp}{m r_p^2}\right)}}{2}$$

Let $r_L^2 = \dfrac{4\hat{T}_\perp}{m\omega_c^2}$

$$\omega_r^\pm = \frac{\omega_c}{2}\left\{1 \pm \sqrt{1 - \left(\frac{2\omega_p^2}{\omega_c^2} + \frac{2r_L^2}{r_p^2}\right)}\right\}$$

d. max density: $1-\left(\frac{2\omega_p^2}{\omega_c^2}+\frac{2r_L^2}{r_p^2}\right)=0$

$$\frac{2\omega_p^2}{\omega_c^2}=1-\frac{2r_L^2}{r_p^2} \qquad \frac{\omega_p^2}{\omega_c^2}=\frac{1}{2}-\frac{r_L^2}{r_p^2}$$

$\omega_r^{\pm}=\frac{\omega_c}{2}$ at the maximum density

e. $U_B=\frac{1}{8}B_0^2 r_p^2$ magnetic field energy per unit length

$$U_E=2\pi\int_0^{r_p} dr\, r\,\frac{|E_r|^2}{8\pi}=\frac{1}{4}\int_0^{r_p} dr\, r\, 4\pi^2 r^2 e^2 \hat{n}^2=\frac{1}{4}\pi^2 e^2 \hat{n}^2 r_p^4$$

$$U_E=\left(\frac{4\pi\hat{n}e^2}{m}\right)^2\cdot\frac{1}{64}\cdot\frac{m^2}{e^2}\cdot r_p^4\cdot\frac{B_0^2}{B_0^2}\cdot\frac{c^2}{c^2}=\frac{\omega_p^2\omega_p^2}{64}\cdot\frac{r_p^4}{\omega_c^2}\frac{B_0^2}{c^2}$$

$$U_E=\frac{1}{8}\frac{\omega_p^2 r_p^2}{c^2}\frac{\omega_p^2}{\omega_c^2}U_B$$

$$U_k=2\pi\int_0^{r_p} dr\, r\, nm\frac{V_\theta^2}{2}=\pi\hat{n}m\omega_r^2\int_0^{r_p} dr\, r^3=\frac{1}{4}\pi\hat{n}m\omega_r^2 r_p^4$$

$$U_k=\frac{4\pi\hat{n}e^2}{m}\cdot\frac{1}{16}\cdot\frac{m^2}{e^2}\cdot\frac{c^2}{B_0^2}\cdot\frac{B_0^2}{c^2}\cdot\omega_r^2 r_p^4=\frac{1}{16}\frac{\omega_p^2}{\omega_c^2}\frac{B_0^2}{c^2}\omega_r^2 r_p^4$$

$$U_k=\frac{1}{2}\frac{\omega_p^2 r_p^2}{c^2}\frac{\omega_r^2}{\omega_c^2}U_B$$