2003 Part I Q7 GPP I/II

a.

Two-fluid equations: (from Freidberg) (MKS)

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha \vec{v}_\alpha) = 0$$

$$m_\alpha n_\alpha \left(\frac{d\vec{v}_\alpha}{dt}\right) = q_\alpha n_\alpha\left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right) - \nabla\cdot\overleftrightarrow{P}_\alpha + \vec{R}_\alpha$$

$$n_\alpha \frac{d}{dt}\left(\frac{m_\alpha v_\alpha^2}{2} + \frac{3}{2}T_\alpha\right) = q_\alpha n_\alpha \vec{v}_\alpha\cdot\vec{E} - \nabla\cdot\left(\vec{v}_\alpha\cdot\overleftrightarrow{P}_\alpha + \vec{q}_\alpha\right) + Q_\alpha + \vec{v}_\alpha\cdot\vec{R}_\alpha$$

Alternate equation for temperature:

$$\frac{3}{2}n_\alpha \frac{dT_\alpha}{dt} = -\overleftrightarrow{P}_\alpha : \nabla\vec{v}_\alpha - \nabla\cdot\vec{q}_\alpha + Q_\alpha$$

where $\overleftrightarrow{P}_\alpha \equiv n_\alpha m_\alpha \langle \vec{w}\vec{w}\rangle$ Pressure tensor

$\vec{w} = \delta\vec{v} = \vec{v} - \langle\vec{v}\rangle$ random thermal motion

$\vec{R}_\alpha \equiv \int m_\alpha \vec{w}\, C_{\alpha\beta}\, d\vec{w}$ Mean momentum transfer between unlike particles due to friction of collisions

$\vec{q}_\alpha = \frac{1}{2} n_\alpha m_\alpha \langle w^2 \vec{w}\rangle$ Heat flux due to random motion

$Q_\alpha = \int \frac{1}{2} m_\alpha W_\alpha^2\, C_{\alpha\beta}\, d\vec{w}$ Heat generated due to collisions between unlike particles

$\vec{q}_\alpha \approx -K_{\parallel\alpha}\nabla_\parallel T_\alpha$ dominated by thermal conductivity along the field

$Q_\alpha \approx n\dfrac{(T_\beta - T_\alpha)}{\tau_{eq}}$ dominated by electron-ion energy equilibration

b. infinite wavelength $\rightarrow$ $\nabla \rightarrow 0$. Looking just at the temperature eqn, with $\nabla \rightarrow k \rightarrow 0$,

$$\frac{3}{2}n_\alpha \frac{dT_\alpha}{dt} = Q_\alpha$$ $\leftarrow$ there can be nontrivial behavior from this

Q_α is already in linear form, so, taking $\frac{d}{dt} \to \frac{\partial}{\partial t} \to -i\omega$, we have

$$-i\omega \frac{3}{2} n \, \delta T_1 = \frac{n}{\tau_{eq}} (\delta T_2 - \delta T_1)$$

$$-i\omega \frac{3}{2} n \, \delta T_2 = \frac{n}{\tau_{eq}} (\delta T_1 - \delta T_2) \qquad \text{assuming } n_1 = n_2.$$

This is a matrix eqn $\Rightarrow$ Take determinant of coefficient matrix equal to 0 to find the frequency (eigenvalue), then calculate eigenvector.

$$\begin{bmatrix} -\frac{3}{2} i\omega + \frac{1}{\tau_{eq}} & -\frac{1}{\tau_{eq}} \\ -\frac{1}{\tau_{eq}} & -\frac{3}{2} i\omega + \frac{1}{\tau_{eq}} \end{bmatrix} \begin{bmatrix} \delta T_1 \\ \delta T_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det = 0 \Rightarrow \left(-\frac{3}{2} i\omega + \frac{1}{\tau_{eq}}\right)^2 = \frac{1}{\tau_{eq}^2}$$

$$\Rightarrow -\frac{3}{2} i\omega + \frac{1}{\tau_{eq}} = \pm \frac{1}{\tau_{eq}}$$

$$\Rightarrow \frac{3}{2} i\omega = \frac{1}{\tau_{eq}} \pm \frac{1}{\tau_{eq}} \qquad \text{Look at the solution with } \omega \neq 0$$

$$i\omega = \frac{4}{3} \frac{1}{\tau_{eq}} \Rightarrow \boxed{\omega = -i \frac{4}{3} \frac{1}{\tau_{eq}}}$$

• negative imaginary part $\rightarrow$ damping

eigenvectors: $\left[-\frac{3}{2} \cdot \left(\frac{4}{3} \frac{1}{\tau_{eq}}\right) + \frac{1}{\tau_{eq}}\right] \delta T_1 = \frac{1}{\tau_{eq}} \delta T_2$

$$\Rightarrow -\delta T_1 = \delta T_2$$

eigenvector $\begin{bmatrix} \delta T_1 \\ \delta T_2 \end{bmatrix} = \delta T \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This mode describes the equilibration of temperatures of two species; it approaches the equilibrium temperature monotonically.

Incidentally, the $\omega = 0$ solution has eigenvector $\begin{bmatrix} \delta T_1 \\ \delta T_2 \end{bmatrix} = \delta T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e., the two temperatures are equal, and nothing happens in time.