May 2001 #2 (EM)

Electromagnetic waves through a plasma

a. $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(\nabla\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{1}{c^2}\left(4\pi\frac{\partial\vec{J}}{\partial t} + \frac{\partial^2\vec{E}}{\partial t^2}\right)$$

- all quantities oscillating as $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
- $\vec{J} = \sum_\sigma n_\sigma q_\sigma \vec{v}_\sigma$
- transverse waves! $\nabla\cdot\vec{E} = \vec{k}\cdot\vec{E} = 0$

$$c^2k^2\vec{E} = i\,4\pi\omega\vec{J} + \omega^2\vec{E}$$

Momentum equation for $\vec{v}_\sigma$: $\quad m_\sigma n_\sigma\left(\frac{\partial\vec{v}_\sigma}{\partial t} + (\vec{v}_\sigma\cdot\nabla)\vec{v}_\sigma\right) = n_\sigma q_\sigma\left(\vec{E} + \frac{\vec{v}_\sigma}{c}\times\vec{B}\right) - \nabla P$

- ignore pressure
- linearize: $n_{\sigma 0} = n_{0}$ (same for electrons and protons: charge neutrality)

$\vec{v}_{\sigma 0} = 0, \quad \vec{E}_0 = 0 \quad \vec{B}_0 = 0$

$m_\sigma n_0 \frac{\partial\vec{v}_\sigma}{\partial t} = n_0 q_\sigma \vec{E} \qquad -i\omega\vec{v}_\sigma = \frac{q_\sigma}{m_\sigma}\vec{E} \qquad \vec{v}_\sigma = i\frac{q_\sigma}{m_\sigma\omega}\vec{E}$

$\vec{J} = \frac{i}{\omega}\sum_\sigma \frac{n_\sigma q_\sigma^2}{m_\sigma}\vec{E}$

$$c^2k^2 = -\sum_\sigma \frac{4\pi n_{\sigma 0} q_\sigma^2}{m_\sigma} + \omega^2$$

$$\frac{c^2k^2}{\omega^2} = 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2} \rightarrow 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi n_e e^2}{m_e}$$

b. $\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2} \sim \frac{1}{m_e} + \frac{1}{m_i}$ $\qquad$ ion term $\frac{1}{m_i}$; very small $\sim\frac{1}{2000}$ correction

c. $\frac{c^2k^2}{\omega^2} \approx 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega^2 = \omega_p^2 + c^2k^2$

$v_\phi = \frac{\omega}{k} = \frac{\sqrt{\omega_p^2 + c^2k^2}}{k} \qquad \frac{c}{v_\phi} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad \boxed{v_\phi = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}}} > c$

$v_g = \frac{d\omega}{dk} \quad 2\omega\frac{d\omega}{dk} = 2c^2k \qquad \omega\frac{d\omega}{dk} = c\cdot\sqrt{\omega^2 - \omega_p^2}$

$$v_g = c\cdot\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c$$

d

e $f_1 = 1660$ MHz $\quad f_2 = 1720$ MHz $\qquad \Delta t = 6.8$ ms

$d = 500$ pc

assume $\omega >> \omega_p$

V_g (speed of pulse): $\quad V_g \approx c\sqrt{1-\frac{\omega_p^2}{\omega^2}} \approx c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)$

$$V_{g1} = c - \frac{c}{2}\frac{\omega_p^2}{\omega_1^2} \qquad V_{g2} = c - \frac{c}{2}\frac{\omega_p^2}{\omega_2^2}$$

$d = V_{g1} t_1 = V_{g2} t_2 \qquad t_1 = \frac{d}{V_{g1}} \quad t_2 = \frac{d}{V_{g2}}$

$\Delta t = t_1 - t_2$

$$\Delta t = d\left(\frac{1}{V_{g1}} - \frac{1}{V_{g2}}\right)$$

$$\frac{\Delta t}{d} = \left(\frac{1}{c\sqrt{1-\frac{\omega_p^2}{\omega_1^2}}} - \frac{1}{c\sqrt{1-\frac{\omega_p^2}{\omega_2^2}}}\right) \approx \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega_1^2} - 1 - \frac{\omega_p^2}{\omega_2^2}\right)$$

$$\frac{c\Delta t}{d} \approx \frac{\omega_p^2}{2}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) = \frac{\omega_p^2}{8\pi^2}\left(\frac{1}{f_1^2} - \frac{1}{f_2^2}\right)$$

$$\omega_p^2 = \frac{c\Delta t}{d} \cdot 8\pi^2 \cdot \frac{f_1^2 f_2^2}{f_2^2 - f_1^2} = \frac{4\pi n e^2}{m_e}$$

$$n_e = \frac{2\pi m_e c \Delta t}{d e^2} \; \frac{f_1^2 f_2^2}{f_2^2 - f_1^2} \qquad f_2^2 - f_1^2 = (f_2 - f_1)(f_2 + f_1)$$

$$n_e \approx \frac{6 \cdot 9 \cdot 10^{-28} \cdot 3 \cdot 10^{10} \cdot 7 \cdot 10^{-3}}{500 \cdot 3 \cdot 10^{18} \cdot 25 \cdot 10^{-20}} \cdot \frac{(1660)^2 \cdot 10^{12} \cdot (1720)^2 \cdot 10^{12}}{10^{12}(1720^2 - 1660^2)}$$

$$\approx \frac{6 \cdot 9 \cdot 7}{5 \cdot 25} \cdot \frac{(1660)^2 \cdot (1720)^2}{(1720 - 1660)(1720 + 1660)} \cdot 10^{-21} \quad 10^{12}$$

$$\approx \frac{9 \cdot 7}{5 \cdot 25 \cdot 10} \cdot \frac{(1660)^2 \cdot (1720)^2}{2 \cdot 1690} \cdot 10^{-9}$$

$n_e \approx \cancel{.13}\ \text{1/cm}^3 \qquad .13\ \text{cm}^{-3}$