2013 Part II Question 4B

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1 Part a

Define the flux as

$$F = \mathrm{Im}\{\psi^* \frac{d\psi}{dx}\}$$

Then the change of flux is

$$\begin{aligned}
\frac{dF}{dx} &= \frac{1}{2i}\frac{d}{dx}\{\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\} \\
&= \frac{1}{2i}\{\psi^* \frac{d^2\psi}{dx^2} + \frac{d\psi^*}{dx}\frac{d\psi}{dx} - \frac{d\psi^*}{dx}\frac{d\psi}{dx} - \psi \frac{d^2\psi^*}{dx^2}\} \\
&= -\frac{1}{2i}\{\psi^* Q(x)\psi - \psi Q^*(x)\psi^*\} = 0
\end{aligned}$$

2 Part b

Concerning the flux change at the pole, we can define $\alpha = 0^-$, $\beta = 0^+$ and look at the flux change between a and b. Note that the α and β are both near zero so the value of ψ can be regarded as a constant.

$$\begin{aligned}
F|_\alpha^\beta &= \mathrm{Im}\{\psi^* \frac{d\psi}{dx}\}|_\alpha^\beta \\
&\approx \mathrm{Im}\{\psi^* \int_\alpha^\beta \frac{d^2\psi}{dx^2}\} \\
&= -\mathrm{Im}\{\psi^* \int_\alpha^\beta \frac{a}{x}\psi\} \\
&\approx -|\psi|^2 \mathrm{Im}\{\int_\alpha^\beta \frac{a}{x}\}
\end{aligned}$$

The integral of a/x across the pole can be separated into a principal value integral and a contour integral surrounding the pole. We can ignore the principal value since it is real. The contour can be chosen to above the pole or below the pole, which gives $-i\pi a$ and $+i\pi a$ respectively.

So if we choose the contour to be above the pole, the flux will increase $a\pi|\psi|^2$ when crossing the pole. If the contour is below the pole, the flux is decreased by that value, which is physical. So the cut should be in the upper plane to make the contour always below the pole.