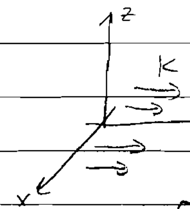
May 1997 #2 (EM)

infinite conducting plane at $z=0$, $\vec{K} = K\hat{y}$ $\quad K = \begin{cases} 0 & t<0 \\ K_0 & t>0 \end{cases}$



$\vec{J} = K\hat{y}\,\delta(z)$

a. $\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}', t-\frac{R}{c})}{R}\,dV'$ $\quad R = |\vec{r}-\vec{r}'|$

- A only has a y-component (as does $\vec{J}$)
- A can only depend on z; $\frac{\partial}{\partial x}=0$, $\frac{\partial}{\partial y}=0$

$\Rightarrow$ observation point at $\rho=0$, $z=z$

$$A_y(z,t) = \frac{1}{c}\int dA' \, \frac{K(t-\frac{R}{c})}{R} \qquad R = |\vec{r}-\vec{r}'| = \sqrt{\rho'^2+z^2}$$

$$A_y(z,t) = \frac{1}{c}\cdot 2\pi \int_0^\infty d\rho'\rho' \, \frac{K(t-\frac{R}{c})}{\sqrt{\rho'^2+z^2}}$$

$$K(t-\tfrac{R}{c}) = K_0\, H(ct-R) = K_0\, H(ct-\sqrt{\rho'^2+z^2}) \qquad H = \text{Heaviside function}$$

if $z>ct$ then $A_y=0$ identically

if $z<ct$, then $H(ct-\sqrt{\rho'^2+z^2})$ is 0 when $\rho' > \sqrt{c^2t^2-z^2}$ (integration limit)

$$A_y(z,t) = \frac{2\pi}{c} H(ct-z)\cdot \int_0^{\sqrt{c^2t^2-z^2}} d\rho' \, \frac{\rho' K_0}{\sqrt{\rho'^2+z^2}} \qquad \begin{aligned} u &= \rho'^2+z^2 \\ du &= 2\rho' d\rho' \\ \tfrac{1}{2}\int u^{-1/2}du &= u^{1/2} = \sqrt{\rho'^2+z^2} \end{aligned}$$

$$\sqrt{\rho'^2+z^2}\Big|_0^{\sqrt{c^2t^2-z^2}} = ct - z$$

$$\boxed{A_y(z,t) = \frac{2\pi K_0}{c}(ct-z)\, H(ct-z)}$$

$$\vec{B} = \nabla\times\vec{A}: \quad B_x = -\frac{\partial}{\partial z}A_y = \frac{-2\pi K_0}{c}\frac{\partial}{\partial z}\left[(ct-z)H(ct-z)\right] = -\frac{2\pi K_0}{c}\left[-H(ct-z) - \cancel{(ct-z)\delta(ct-z)}\right]$$

$$\boxed{B_x(z,t) = \frac{2\pi K_0}{c} H(ct-z)}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad E_y = -\frac{1}{c}\frac{\partial}{\partial t}\frac{2\pi K_0}{c}(ct-z)H(ct-z)$$

$$E_y = \frac{-1}{c}\frac{2\pi K_0}{c}\left[cH(ct-z) + \cancel{(ct-z)c\,\delta(ct-z)}\right]$$

$$E_y = -\frac{2\pi K_0}{c}H(ct-z)$$

b. $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}E_yB_x\,\hat{y}\times\hat{x} = -\frac{c}{4\pi}\hat{z}\,E_yB_x =$

$$\vec{S} = -\hat{z}\frac{c}{4\pi}\cdot\frac{-2\pi K_0}{c}\cdot\frac{2\pi K_0}{c}H(ct-z)$$

$$\vec{S} = \hat{z}\,\frac{\pi K_0^2}{c}H(ct-z)$$

Total power radiated per unit area

$= \frac{\pi K_0^2}{c}\cdot 2$ radiation above + below the plane

$S = \frac{\text{energy}}{\text{time}\cdot\text{area}}$; $\lim_{t\to\infty}\frac{1}{t}\int_0^t dt'\,H(ct'-z) = 1$