Generals Exam 2005, Part II, Problem 1

MHD Energy Conservation

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The following identity is given at the beginning of the problem and is used throughout the calculation. For a scalar function $f(\mathbf{r}, t)$, define

$$F(t) \equiv \int_V d^3\mathbf{r}\, f(\mathbf{r}, t), \tag{1}$$

where V denotes the volume of integration. Then

$$\frac{dF}{dt} = \int_V d^3\mathbf{r}\, \frac{\partial f}{\partial t} + \int_S d^2\mathbf{r}\, \hat{\mathbf{n}} \cdot \mathbf{u}\, f, \tag{2}$$

where S is the bouding surface of V, $\hat{\mathbf{n}}$ is the unit normal vector of this surface pointing outward, and $\mathbf{u}$ is the local velocity of the boundary.

Part (a)

With the MHD energy density given by

$$w_p = \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}, \tag{3}$$

use the ideal MHD equations to find

$$\frac{dW_p}{dt} = \frac{d}{dt}\int_{V_p} d^3\mathbf{r}\, w_p = \int_{V_p} d^3\mathbf{r}\, \frac{\partial w_p}{\partial t} + \int_{S_p} d^2\mathbf{r}\, \hat{\mathbf{n}}_p \cdot \mathbf{u}_p\, w_p, \tag{4}$$

First compute $\partial w_p/\partial t$:

$$\frac{\partial w_p}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right] = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

Subsitute for $\partial\rho/\partial t$ according to the continuity equation, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{6}$$

Also substitute for $\partial\mathbf{B}/\partial t$ using Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

and for $\partial\mathbf{v}/\partial t$ using the momentum equation with $p = 0$:

$$\rho\frac{d\mathbf{v}}{dt} = \rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = \mathbf{J} \times \mathbf{B}. \tag{8}$$

Now $\partial w_p/\partial t$ is given by

$$\frac{\partial w_p}{\partial t} = -\frac{1}{2}v^2\nabla \cdot (\rho\mathbf{v}) + \rho\mathbf{v} \cdot \left[\frac{\mathbf{J} \times \mathbf{B}}{\rho} - (\mathbf{v} \cdot \nabla)\mathbf{v}\right] - \frac{1}{\mu_0}\mathbf{B} \cdot (\nabla \times \mathbf{E})\,. \tag{9}$$

Noting that

$$\begin{aligned} \nabla \cdot (\rho v^2 \mathbf{v}) &= v^2 \nabla \cdot (\rho \mathbf{v}) + \rho \mathbf{v} \cdot \nabla v^2 \\ (\mathbf{v} \cdot \nabla)\mathbf{v} &= \tfrac{1}{2} \nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}), \end{aligned} \tag{10}$$

Equation 9 becomes

$$\frac{\partial w_p}{\partial t} = -\frac{1}{2}\nabla \cdot (\rho v^2 \mathbf{v}) + \cancel{\frac{1}{2}\rho \mathbf{v} \cdot \nabla v^2} + \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) - \cancel{\frac{1}{2}\rho \mathbf{v} \cdot \nabla v^2} + \cancel{\rho \mathbf{v} \cdot \left[\mathbf{v} \times (\nabla \times \mathbf{v})\right]} - \frac{1}{\mu_0} \mathbf{B} \cdot (\nabla \times \mathbf{E}) \,. \tag{11}$$

Rearranging the $\mathbf{J} \times \mathbf{B}$ term and substituting for $\mathbf{J}$ from Ampère's Law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) gives

$$\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) = -(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{J} = -\frac{1}{\mu_0}(\mathbf{v} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) = \frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}), \tag{12}$$

where the final expression is obtained by substituting for $\mathbf{v} \times \mathbf{B}$ from Ohm's Law ($\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$). Plugging this expression back into Equation 11 gives

$$\frac{\partial w_p}{\partial t} = -\frac{1}{2}\nabla \cdot (\rho v^2 \mathbf{v}) + \frac{1}{\mu_0}\Big[\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E})\Big]. \tag{13}$$

Noting that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{14}$$

gives the final expression for $\partial w_p / \partial t$ to be

$$\frac{\partial w_p}{\partial t} = -\frac{1}{2}\nabla \cdot (\rho v^2 \mathbf{v}) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \nabla \cdot \left[-\frac{1}{2}\rho v^2 \mathbf{v} - \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right]. \tag{15}$$

Substituting the above expression and w_p from Equation 3 into Equation 4 gives

$$\frac{dW_p}{dt} = \int_{V_p} d^3\mathbf{r}\, \nabla \cdot \left[-\frac{1}{2}\rho v^2 \mathbf{v} - \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right] + \int_{S_p} d^2\mathbf{r}\, \hat{\mathbf{n}}_p \cdot \mathbf{u}_p \left[\frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right] \tag{16}$$

Using the Divergence Theorem to convert the volume integral over V_p to a surface integral over S_p and asserting that the velocity of S_p is the plasma fluid velocity at the surface such that $\mathbf{u}_p = \mathbf{v}$, the above equation can be rewritten as

$$\frac{dW_p}{dt} = \int_{S_p} d^2\mathbf{r}\, \hat{\mathbf{n}}_p \cdot \left[-\cancel{\frac{1}{2}\rho v^2}\mathbf{v} - \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} + \cancel{\frac{1}{2}\rho v^2}\mathbf{v} + \mathbf{v}\frac{B^2}{2\mu_0}\right], \tag{17}$$

which gives the final rate of change for the total MHD energy to be

$$\frac{dW_p}{dt} = \int_{S_p} d^2\mathbf{r}\, \hat{\mathbf{n}}_p \cdot \left[-\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} + \mathbf{v}\frac{B^2}{2\mu_0}\right] \tag{18}$$

Part (b)

Now consider the vacuum energy density, which is given by

$$w_v = \frac{B^2}{2\mu_0}. \tag{19}$$

Again look for the rate of change of the total energy of the form

$$\frac{dW_v}{dt} = \frac{d}{dt}\int_{V_v} d^3\mathbf{r}\, w_v = \int_{V_v} d^3\mathbf{r}\, \frac{\partial w_v}{\partial t} + \int_{S_v} d^2\mathbf{r}\, \hat{\mathbf{n}}_v \cdot \mathbf{u}_v\, w_v, \tag{20}$$

where the vacuum volume V_v is the volume between the plasma and the conducting wall. As before, compute $\partial w_v/\partial t$:

$$\frac{\partial w_v}{\partial t} = \frac{\partial}{\partial t}\left[\frac{B^2}{2\mu_0}\right] = \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} \tag{21}$$

Substituting from Faraday's Law (Equation 7) and from the vector identity in Equation 14, the above expression for $\partial w_v/\partial t$ can be rewritten as

$$\frac{\partial w_v}{\partial t} = -\frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E}) = -\frac{1}{\mu_0}\left[\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \cancel{\mathbf{E}\cdot(\nabla\times\mathbf{B})}\right] = -\nabla\cdot\left[\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right]. \tag{22}$$

The $\nabla\times\mathbf{B}$ term can be cancelled because $\mathbf{J}=0$ in the vacuum region such that Ampère's Law becomes $\nabla\times\mathbf{B}=0$. Inserting the above expression into Equation 20 along with w_v from Equation 19 gives

$$\frac{dW_v}{dt} = \int_{V_v} d^3\mathbf{r}\,\nabla\cdot\left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right] + \int_{S_v} d^2\mathbf{r}\,\hat{\mathbf{n}}_v\cdot\mathbf{u}_v\left[\frac{B^2}{2\mu_0}\right] \tag{23}$$

Again converting the volume integral using the Divergence theorem gives the final expression for the rate of change of the total vacuum energy to be

$$\frac{dW_v}{dt} = \int_{S_v} d^2\mathbf{r}\,\hat{\mathbf{n}}_v\cdot\left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{u}_v\frac{B^2}{2\mu_0}\right] \tag{24}$$

Part (c)

The jump conditions at the plasma-vacuum interface are that

$$\mathbf{u}_p = \mathbf{v} \quad \text{and} \quad \mathbf{J} = \begin{cases} \mathbf{J}, & \text{inside} \\ 0, & \text{outside.} \end{cases} \tag{25}$$

It is very possible that more stringent jump conditions are required if the surface current of the plasma on the boundary surface S_p is considered. Is this extra consideration necessary to obtain a complete solution to the problem?

The rate of change of the total energy is given by

$$\frac{dW}{dt} = \frac{dW_p}{dt} + \frac{dW_v}{dt}. \tag{26}$$

To relate dW_v/dt to dW_p/dt, examine the structure vacuum bounding surface S_v. This bounding surface has two distinct parts: the inner surface, which is defined by the plasma boundary S_p, and the outer surface, which is defined by the conductor boundary S_c. Because the unit normal to the inner surface points out of the vacuum volume and into the plasma, it is given by $\mathbf{n}_v = -\mathbf{n}_p$ on the surface S_p. Likewise, the unit normal to the outer surface points out of the vacuum volume and into the conductor, so $\mathbf{n}_v = -\mathbf{n}_c$ on the surface S_c. Rewriting dW_v/dt from Equation 24 in terms of separate integrations over S_p and S_c gives

$$\frac{dW_v}{dt} = -\int_{S_p} d^2\mathbf{r}\,\hat{\mathbf{n}}_p \cdot \left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{u}_p\frac{B^2}{2\mu_0}\right] - \int_{S_c} d^2\mathbf{r}\,\hat{\mathbf{n}}_c \cdot \left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{u}_c\frac{B^2}{2\mu_0}\right] \tag{27}$$

The boundary conditions at the conducting wall are

$$\mathbf{u}_c = 0 \quad \text{and} \quad \mathbf{E} \parallel \hat{\mathbf{n}}_c. \tag{28}$$

The first condition is valid because the conductor is a fixed object, and the second is a physical consequence of the perfect conductivity of the wall. With $\mathbf{E} \parallel \hat{\mathbf{n}}_c$, the first term of the S_c integration will vanish because $\hat{\mathbf{n}}_c\cdot(\mathbf{E}\times\mathbf{B}) = 0$, and with $\mathbf{u}_c = 0$, the second term vanishes as well. Consequently, the contribution of the integration over S_c vanishes entirely. This leaves

$$\frac{dW_v}{dt} = -\int_{S_p} d^2\mathbf{r}\,\hat{\mathbf{n}}_p \cdot \left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{v}\frac{B^2}{2\mu_0}\right] \tag{29}$$

Substituting Equations 18 and 29 into Equation 26 gives

$$\frac{dW}{dt} = \cancel{\int_{S_p} d^2\mathbf{r}\,\hat{\mathbf{n}}_p \cdot \left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{v}\frac{B^2}{2\mu_0}\right]} - \cancel{\int_{S_p} d^2\mathbf{r}\,\hat{\mathbf{n}}_p \cdot \left[-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \mathbf{v}\frac{B^2}{2\mu_0}\right]} = 0 \tag{30}$$

With $dW/dt = 0$, the total energy in the system is conserved.

Part (d)

Defining the respective electromagnetic field energy densities to be

$$w_E \equiv \frac{\epsilon_0 E^2}{2} \quad \text{and} \quad w_B \equiv \frac{B^2}{2\mu_0}, \tag{31}$$

the ratio of the two densities will be

$$\frac{w_E}{w_B} = \epsilon_0\mu_0\frac{E^2}{B^2} = \frac{1}{c^2}\frac{E^2}{B^2} \tag{32}$$

According to Ohm's Law, $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$ such that

$$E^2 = (\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{B}) = v^2B^2 - (\mathbf{v}\cdot\mathbf{B})^2. \tag{33}$$

This expression is at a maximum when $\mathbf{v} \perp \mathbf{B}$ such that

$$\left(E^2\right)_{max} = v^2B^2. \tag{34}$$

Thus, the maximum ratio of electromagnetic field energies will be

$$\left(\frac{w_E}{w_B}\right)_{max} = \frac{1}{c^2}\frac{v^2\cancel{B^2}}{\cancel{B^2}} = \frac{v^2}{c^2} \tag{35}$$

Because velocities in ideal MHD are assumed to be non-relativistic, $v \ll c$ and

$$\left(\frac{w_E}{w_B}\right)_{max} \ll 1, \tag{36}$$

so the electric field energy density w_E is negligible compared to the magnetic field energy density w_B. It therefore can be neglected in the expression for w_p given in Equation 3.