May 2000 #1 (QM)

Spin $\frac{1}{2}$ particle in 1D

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \omega S_z$

eigenstates: $|\psi\rangle = |n, m\rangle$ $\quad n = 0, 1, 2, \ldots \quad m = -\frac{1}{2}, \frac{1}{2}$

$E = (n+\frac{1}{2})\hbar\omega + m\hbar\omega$

$E = 0, \quad \hbar\omega, \quad \hbar\omega, \quad 2\hbar\omega, \quad 2\hbar\omega, \ldots$

$|0,-\frac{1}{2}\rangle \quad |0,\frac{1}{2}\rangle \quad |1,-\frac{1}{2}\rangle \quad |1,\frac{1}{2}\rangle \quad |2,-\frac{1}{2}\rangle \ldots$

$H' = \alpha x S_x$

For the nondegenerate state, $E' = \langle 0,-\frac{1}{2}|H'|0,-\frac{1}{2}\rangle = \alpha\langle 0,-\frac{1}{2}|xS_x|0,-\frac{1}{2}\rangle$

$S_x = \frac{\hbar}{2}\sigma_x \rightarrow \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\quad \sigma_x|-\frac{1}{2}\rangle \Rightarrow \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Longrightarrow |\frac{1}{2}\rangle$

$E' = 0$

Degenerate states $\rightarrow$ degenerate perturbation theory

- For each degenerate subspace, diagonalize H' to find the correct "stable" basis to start from

In a given degenerate subspace, two eigenstates are $|n,\frac{1}{2}\rangle$ and $|n+1,-\frac{1}{2}\rangle$

Matrix elements of H':

$$\begin{array}{c|cc} nm & n,\frac{1}{2} & n+1,-\frac{1}{2} \\ \hline n,\frac{1}{2} & & \\ n+1,-\frac{1}{2} & & \end{array}$$

$(H')_{11} = \langle n,\frac{1}{2}|H'|n,\frac{1}{2}\rangle$

$(H')_{12} = \langle n,\frac{1}{2}|H'|n+1,-\frac{1}{2}\rangle$

$(H')_{21} = \langle n+1,-\frac{1}{2}|H'|n,\frac{1}{2}\rangle = (H')_{12}^*$

$(H')_{22} = \langle n+1,-\frac{1}{2}|H'|n+1,-\frac{1}{2}\rangle$

$\langle n,\frac{1}{2}|H'|n,\frac{1}{2}\rangle = \alpha\frac{\hbar}{2}\langle n,\frac{1}{2}|x\sigma_x|n,\frac{1}{2}\rangle = 0$

$\langle n,\frac{1}{2}|H'|n+1,-\frac{1}{2}\rangle = \alpha\frac{\hbar}{2}\langle n,\frac{1}{2}|x\sigma_x|n+1,-\frac{1}{2}\rangle = \alpha\frac{\hbar}{2}\langle n,\frac{1}{2}|x|n+1,\frac{1}{2}\rangle$

In the harmonic oscillator system, $x = \frac{x_0}{\sqrt{2}}(a + a^+)$ $\quad x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$

And $a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle$

$\langle n,\frac{1}{2}|H'|n+1,-\frac{1}{2}\rangle = \frac{\alpha\hbar x_0}{2\sqrt{2}}\langle n,\frac{1}{2}|\left(\sqrt{n+1}|n,\frac{1}{2}\rangle + \sqrt{n+2}|n+2,\frac{1}{2}\rangle\right)$

$= \frac{\alpha\hbar x_0\sqrt{n+1}}{2\sqrt{2}} = \alpha\sqrt{\frac{(n+1)\hbar^3}{8m\omega}} \equiv \Delta_n$

$\langle n+1,-\frac{1}{2}|H'|n+1,-\frac{1}{2}\rangle = 0$

For this subspace, $H' = \begin{bmatrix} 0 & \Delta_n \\ \Delta_n & 0 \end{bmatrix}$ (Proportional to Pauli Matrix σ_x)

$\rightarrow$ Eigenvalues are $\pm \Delta_n$, with eigenstates

$$\frac{1}{\sqrt{2}}\left[|n, \tfrac{1}{2}\rangle \pm |n+1, -\tfrac{1}{2}\rangle \right]$$

E' = Energy Level Shift = $\langle n^0 | H' | n^0 \rangle = \pm\Delta = \cancel{\propto} \sqrt{\frac{(n+1)\hbar^3}{8m\omega}}$

for the degenerate states