2002 Part II Q3 — Asymptotics

$$I(x) = \int_0^1 (1+t)\, e^{x\sin(3t)}\, dt \qquad x \to \infty$$

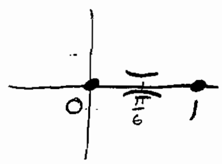
$\phi = x\sin(3t)$

$\phi' = 3x\cos(3t)$ saddles at $3t_s = \frac{\pi}{2} + n\pi$ $\quad t_s = \frac{\pi}{6} \quad t_s = \frac{\pi}{2}$ $\quad \frac{\pi}{2} > 1$ though

$\phi'' = -9x\sin(3t)$

at $t_s = \frac{\pi}{6}$, $\phi''(\frac{\pi}{6}) = -9x\sin(\frac{\pi}{2}) = -9x$

$\phi''\Delta t^2 = -\text{real} \qquad -9\Delta t^2 = -1 \qquad \Delta t^2 > 0$



At $t=0$, $\phi' = 3x$: direction of steepest descent is to the left

At $t=1$, $\phi' = 3x\cos(3)$: direction of steepest descent is to the right

$\Rightarrow$ no endpoint contribution, only saddle

$$(1+\tfrac{\pi}{6}) \int_{t_s-\infty}^{t_s+\infty} dt\; e^{\phi} \sim 2(1+\tfrac{\pi}{6}) e^{\phi_s} \int_{t_s}^{t_s+\infty} dt\, e^{\frac{1}{2}\phi_s''(t-t_s)^2}$$

$$u = -\tfrac{1}{2}\phi_s''(t-t_s)^2 = \tfrac{9}{2}x(t-t_s)^2 \qquad du = 9x(t-t_s)\,dt$$

$$(t-t_s) = \sqrt{\frac{2u}{9x}} \qquad du = \sqrt{9x}\sqrt{2u}\,dt$$

$$I \sim 2(1+\tfrac{\pi}{6}) e^{\phi_s} \int_0^\infty \frac{du}{\sqrt{9x}\sqrt{2}\sqrt{u}}\, e^{-u}$$

$$I \sim \frac{2(1+\frac{\pi}{6}) e^{\phi_s}}{\sqrt{18x}} \cdot \sqrt{\pi} \qquad e^{\phi_s} = e^{x}$$

$$I \sim \sqrt{\frac{2\pi}{9x}} \left(1+\frac{\pi}{6}\right) e^{x}$$