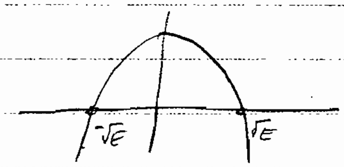
2003 Part II Q5

Asymptotics

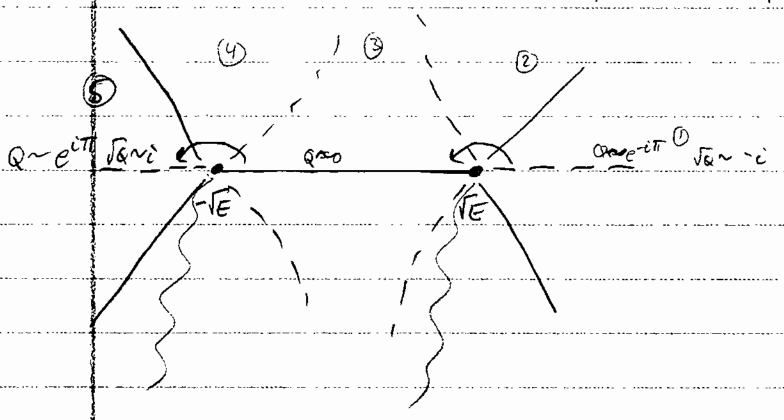
$y'' + (E - x^2)y = 0 \qquad y(a) = y(-a) = 0 \qquad E > 0$

a. Stokes plot. $Q = E - x^2$



$Q' = -2x$

$\frac{Q'}{Q^{3/2}} = \frac{-2x}{(E-x^2)^{3/2}} << 1$ for WKB validity



$z_0 = \sqrt{E}$

b. values of E for $a \to \infty$

Bohr – Sommerfeld condition

Region ①: choose the subdominant solution

$(z_0, z) = \frac{1}{Q^{1/4}} e^{i\int_{z_0}^{z} \sqrt{Q}\, dz} \sim e^{\int_{z_0}^{z} \sqrt{|Q|}\, dz}$ which is dominant

① $(z, z_0)_s$

② $(z, z_0)_d$

③ $(z, z_0)_d + i(z_0, z)_s$

Reconnect: $(z, -z_0)_d [-z_0, z_0] + i[z_0, -z_0](-z_0, z)_s$

$[-z_0, z_0] = e^{i\int_{-z_0}^{z_0} \sqrt{Q}\, dz} \equiv e^{iW} \qquad W = \int_{-z_0}^{z_0} \sqrt{Q}\, dz$

$e^{iW}(z, -z_0)_d + i e^{-iW}(-z_0, z)_s$

④ $e^{iW}(z, -z_0)_d + i[e^{iW} + e^{-iW}](-z_0, z)_s$

⑤ $e^{iW}(z, -z_0)_s + i[e^{iW} + e^{-iW}](-z_0, z)_d$

Require the coefficient of the dominant piece to be 0 for $y \to 0$ at $x \to -\infty$

$\Rightarrow \quad e^{iW} + e^{-iW} = 0 \qquad e^{i2W} + 1 = 0 \qquad 2W = \pi + 2n\pi$

$W = (n + \tfrac{1}{2})\pi$

$$\Rightarrow \int_{-z_0}^{z_0} \sqrt{Q}\, d\cancel{x} = (n+\tfrac{1}{2})\pi \qquad \sqrt{Q} = \sqrt{E-x^2}$$

$$\int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E-x^2}\, dx = \sqrt{E}\int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{1-\tfrac{x^2}{E}}\, dx \qquad \text{let } u = \frac{x}{\sqrt{E}} \qquad du = \frac{dx}{\sqrt{E}}$$

$$= E\int_{-1}^{1} du\, \sqrt{1-u^2} = E\frac{\pi}{2}$$

$$E\frac{\pi}{2} = (n+\tfrac{1}{2})\pi$$

$$E = 2n+1 \qquad n = 0,1,2,\ldots$$

$$E = 1, 3, 5, 7, \ldots$$

c. lowest value of E, when a is finite. (assume $a > \sqrt{E}$)

Solution will be even in x, which means that its derivative will be 0 at $x = 0$.

Two conditions: $y(a) = 0$, $y'(0) = 0$

On the real line, now write $y = (z,z_0)_s + C(z_0,z)_d$ $(C \sim \text{small})$

① $y = [1+\frac{iC}{2}](z,z_0)_s + C(z_0,z)_d$ picking up half the Stokes constant, stepping off the real line

② $y = [1+\frac{iC}{2}](z,z_0)_d + c(z_0,z)_s$

③ $y = [1+\frac{iC}{2}](z,z_0)_d + (i+c-\frac{c}{2})(z_0,z)_s$

$\quad = [1+\frac{iC}{2}](z,z_0)_d + (i+\frac{c}{2})(z_0,z)_s$

BCs: $(0,z_0)_s + c(z_0,a)_d = 0$ on the real line

$[1+\frac{i}{2}](z,z_0)'|_0 + (i+c-\frac{1}{2})(z_0,z)'|_0 = 0$

Region ① $(z,z_0) = Q^{-1/4} e^{-i\int_{z_0}^{z} \sqrt{Q}\, dx}$ $\quad \sqrt{Q} \sim -i$, let $\int_{z_0}^{z} \sqrt{|Q|}\, dx = f(z)$

$\rightarrow (z,z_0) = Q^{-1/4} e^{-f} \qquad (z_0,z) = Q^{-1/4} e^{f}$

$$BC \Rightarrow Q(a)^{-1/4}\left[e^{-f(a)} + c\, e^{f(a)}\right] = 0$$

$$\Rightarrow e^{-f(a)} + c\, e^{f(a)} = 0$$

$$\Rightarrow c = -e^{-2f(a)}$$

$f(a)$ can be computed:

$$f(a) = \int_{\sqrt{E}}^{a} \sqrt{x^2 - E}\, dx$$ can be done with substitution $x = E\cosh u$

$$\Rightarrow \quad f(a) = \frac{1}{2} E \left(\cosh^{-1} \frac{a}{\sqrt{E}} + \frac{a}{\sqrt{E}} \sqrt{\frac{a^2}{E} - 1} \right)$$

If $a >> \sqrt{E}$, then c is exponentially small

Region ③: $(z, z_0)_d = \dfrac{1}{Q^{1/4}} e^{i\int_z^{z_0} \sqrt{Q}\, dx}$ $\quad Q \sim e^{i0}$ $\quad (z_0, z) = \dfrac{1}{Q^{1/4}} e^{-i\int_z^{z_0} \sqrt{Q}\, dx}$

$$\Rightarrow \quad (z, z_0)' = \frac{-\frac{1}{4} Q'(z)}{Q^{5/4}} e^{i\int_z^{z_0} \sqrt{Q}\, dx} - i Q^{1/4} e^{i\int_z^{z_0} \sqrt{Q}\, dx}$$

$$(z_0, z)' = \frac{-\frac{1}{4} Q'(z)}{Q^{5/4}} e^{-i\int_z^{z_0} \sqrt{Q}\, dx} + i Q^{1/4} e^{-i\int_z^{z_0} \sqrt{Q}\, dx}$$

But $Q' = -2x$, so $Q'(0) = 0$ and $Q(0) = E$

$$\Rightarrow \quad (z, z_0)'|_0 = -iE^{1/4} e^{i\int_0^{z_0} \sqrt{Q}\, dx}$$
$$(z_0, z)'|_0 = iE^{1/4} e^{-i\int_0^{z_0} \sqrt{Q}\, dx}$$

Note $\int_0^{z_0} \sqrt{Q}\, dx = \frac{1}{2} W$ from before $= \frac{1}{2}\left(\frac{E\pi}{2}\right) = \frac{\pi}{4} E$ $\quad \left(W = \frac{\pi}{2} E\right)$

$$BC: \quad -iE^{1/4} \left[\left(1 + \frac{ic}{2}\right) e^{\frac{i}{2}W} - \left(i + \frac{c}{2}\right) e^{-\frac{i}{2}W} \right] = 0$$

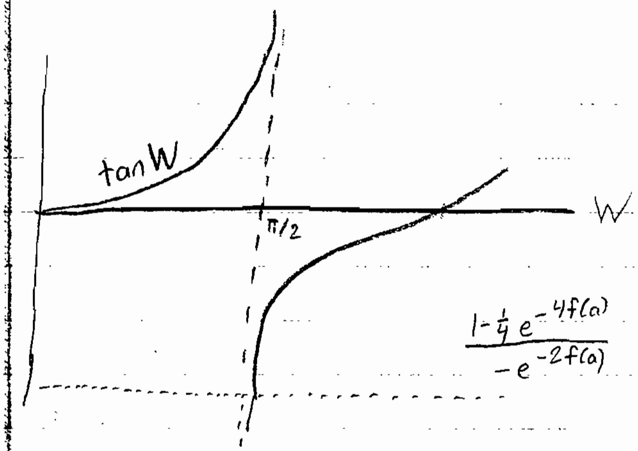
$$\left(1 + \frac{ic}{2}\right) e^{\frac{i}{2}W} = \left(i + \frac{c}{2}\right) e^{-\frac{i}{2}W}$$

$$e^{iW} = \frac{i + \frac{c}{2}}{1 + \frac{ic}{2}} \qquad e^{iW} = \frac{c + i\left(1 - \frac{c^2}{4}\right)}{1 + c^2/4}$$

$$\Rightarrow \cos W = \frac{c}{1 + c^2/4} \qquad \sin W = \frac{1 - c^2/4}{1 + c^2/4}$$

$$\Rightarrow \quad \tan W = \frac{1 - c^2/4}{c}$$ substitute $c = -e^{-2f(a)}$

$$\tan W = \frac{1 - \frac{1}{4} e^{-4f(a)}}{-e^{-2f(a)}}$$



$$\frac{1-\frac{1}{4}e^{-4f(a)}}{-e^{-2f(a)}} \leftarrow \text{ for large } a, \text{ treating } E \text{ as fixed}$$

$\tan W = \tan\frac{\pi}{2}E.$ If $a \to \infty$, $f(a) \to \infty$, $\tan W \to -\infty$, $W \to \frac{\pi}{2} \Rightarrow E = 1$ as before

But $f(a)$ is finite and depends on E

$$\Rightarrow \tan\frac{\pi}{2}E = \frac{1-\frac{1}{4}\exp\left[-2E\left(\cosh^{-1}\frac{a}{\sqrt{E}} + \frac{a}{\sqrt{E}}\sqrt{\frac{a^2}{E}-1}\right)\right]}{-\exp\left[-E\left(\cosh^{-1}\frac{a}{\sqrt{E}} + \frac{a}{\sqrt{E}}\sqrt{\frac{a^2}{E}-1}\right)\right]}$$

$\boxed{E>1}$ for the lowest