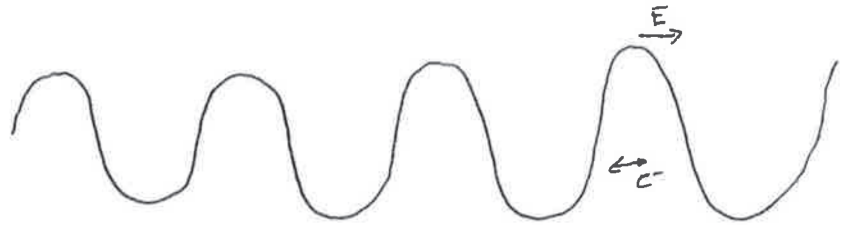
$E = -\nabla\phi$

$V \sim e\phi$

a)



$E = \frac{1}{2}mv_0^2 + e\phi = \frac{1}{2}mv^2$

$w(\phi, v_0) =$

So we're moving with the frame of the wave's phase velocity, and considering the motion of a particle as it interacts with this ϕ.

Let w be the particle velocity in K'. then w and v are related by

$$W = V - V_{ph} = V - \omega/K.$$

By conservation of energy, $\frac{1}{2}mW^2 = \frac{1}{2}mW_0^2 + e\phi$

$$W^2 = W_0^2 + \frac{2e\phi}{m}$$

$$\boxed{W = \left[W_0^2 + \frac{2e\phi}{m}\right]^{1/2}}$$

b) $|W|dn = |W_0| n_0 f_w(W_0) dW_0$

total charge density: $n_e(x) = \int \frac{|W_0| n_0 f_w(W_0)}{|W|} dW_0$

Poisson: $-\nabla^2\phi = 4\pi e(n_i - n_e) = 4\pi e n_0 \left(1 - \int \frac{|W_0| f_w(W_0)}{|W|} dW_0\right)$

$$k^2\phi = 4\pi e n_0 \left[1 - \int \frac{|W_0| f_w(W_0)}{|W|} dW_0\right] = n - n_0$$

$$k\phi - \frac{4\pi e n_0}{k^2}\left[1 - \int \frac{|W_0| f_w(W_0)}{\left[W_0^2 + \frac{2e\phi}{m}\right]^{1/2}} dW\right] = 0$$

$$\frac{|W_0|}{\left[W_0^2 + \frac{2e\phi}{m}\right]^{1/2}} = \frac{1}{\left[1 + \frac{2e\phi}{mW_0^2}\right]^{1/2}}$$

$\sqrt{1+x} \approx 1 + \frac{1}{2}x$ $\quad\left(\frac{2e\phi}{m} \ll W_0^2\right)$

$$\approx \frac{1}{1 + \frac{e\phi}{mW_0^2}}$$

$\frac{1}{1-x} \approx 1 + x$

1 Cont.

From $\phi - \frac{4\pi e n_0}{k^2}\left[1 - \int \frac{|w_0| f_w(w_0)}{\left[w_0^2 + \frac{2e\phi}{m}\right]^{1/2}} dw\right] = 0$

$$\frac{|w|}{\left[w_0^2 + \frac{2e\phi}{m}\right]^{1/2}} = \frac{1}{\left[1 + \frac{2e\phi}{v_0^2 m}\right]^{1/2}} \approx \frac{1}{1 + \frac{e\phi}{w_0^2 m}} \approx 1 - \frac{e\phi}{w_0^2 m}$$

$$\phi - \frac{4\pi e n_0}{k^2}\left[1 - \int f_w(w_0)\,dw + \frac{e\phi}{m}\int \frac{f_w(w_0)}{w_0^2} dw\right] = 0$$

$$1 - \frac{4\pi e^2 n_0}{m k^2}\int \frac{f_{w_0}}{w_0^2} dw = 0$$

Now go back to v frame:
$w_0 = v - \omega/k$

$$\boxed{1 - \frac{\omega_p^2}{k^2}\int \frac{f_{w_0}}{(v - \omega/k)^2} dv = 0}$$

c) $1 - \frac{\omega_p^2}{k^2}\int \frac{f_{0}(v)}{\left(\frac{\omega}{k}\right)^2\left(1 - \frac{kv}{\omega}\right)^2} = 1 - \frac{\omega_p^2}{\omega^2}\int \frac{f_0(v)}{\left(1 - \frac{kv}{\omega}\right)^2} = 0$

$$\frac{1}{\left(1 - \frac{kv}{\omega}\right)^2} \approx 1 + 2\left(\frac{kv}{\omega}\right) + 3\left(\frac{kv}{\omega}\right)^2$$

$1 - \frac{\omega_p^2}{\omega^2} + \frac{3\omega_p^2 k^2}{\omega^4}\int v^2 f_0(v)\,dv = 0$ Assume Maxwellian $f_0 = \frac{1}{\sqrt{2\pi}\,v_{th}} e^{-v^2/2v_{th}^2}$

$$\int v^2 f_0(v)\,dv = \frac{1}{\sqrt{2\pi}\,v_{th}}\int v^2 e^{-v^2/2v_{th}^2} dv$$

$$\int v^2 e^{-v^2/2v_{th}^2} dv = -v_{th}^2 \int v \frac{d}{dv}\left[e^{-v^2/2v_{th}^2}\right] dv.$$

, cont) .

$$-\cancel{2}v_{tn}^2 \int_{-\infty}^{\infty} v \frac{d}{dv}\left[e^{-v^2/2v_{tn}^2}\right] dv \overset{IBP}{=} \cancel{2}v_{tn}^2 \int_{-\infty}^{\infty} e^{-v^2/2v_{tn}^2} dv =$$

$$\cancel{2}v_{tn}^2 \sqrt{2\pi}\, v_{tn}$$

so $1 - \frac{\omega_p^2}{\omega^2} + \frac{3\omega_p^2 k^2}{\omega^4}\int v^2 f_0(v)\, dv = 0 \Rightarrow$

$$1 - \frac{\omega_p^2}{\omega^2} + \frac{3\omega_p^2 k^2}{\omega^4}\left[\frac{\cancel{2} v_{tn}^2 \cancel{\sqrt{2\pi}}\, \cancel{v_{tn}}}{\cancel{\sqrt{2\pi}}\, \cancel{v_{tn}}}\right] = 1 - \frac{\omega_p^2}{\omega^2} + \frac{3\omega_p^2 k^2 v_{tn}^2}{\omega^4} = 0$$

$$\boxed{\omega^2 - \omega_p^2 \mp \frac{3\omega_p^2 k^2 v_{tn}^2}{\omega^2}} = 0$$