2006 Part II Q4 Asymptotics

a. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$ asymptotic behavior as $x \to \infty$

$y = e^S \quad S'' + (S')^2 + xS' - x = 0$

$(S')^2 + xS' = 0 \quad S' + x = 0 \quad S' = -x$

$S' = -x + g' \rightarrow S'' = -1 + g'' \quad (S')^2 = x^2 - 2xg' + g'^2$

$\Rightarrow \overset{\sim 0}{\cancel{-1}} + \overset{\sim 0}{\cancel{g''}} + \cancel{x^2} - 2xg' + g'^2 - \cancel{x^2} + xg' - x = 0$

$-xg' + g'^2 - x = 0 \quad g' = -1$

$S' = -x - 1 + g' \quad S'' = -1 + g'' \quad (S')^2 = x^2 + 2x - 2xg' + 1 - 2g' + (g')^2$

$\Rightarrow \cancel{-1} + g'' + \cancel{x^2} + \cancel{2x} - 2xg' + \cancel{1} - 2g' + (g')^2 - \cancel{x^2} - \cancel{x} + g'x - \cancel{x} = 0$

$g'' - xg' - \overset{\sim 0}{2g'} + (g')^2 = 0$ no dominant balance? $\Rightarrow$ <u>$g' = 0$ solves it</u>

$s' = -x - 1 \quad S = -\frac{1}{2}x^2 - x$

$\boxed{y = e^{-\frac{1}{2}x^2 - x}}$ exact solution

$xS' = x \quad S' = 1$

$S' = 1 + g' \quad S'' = g'' \quad (S')^2 = 1 + 2g' + (g')^2$

$g'' + 1 + \overset{\sim 0}{2g'} + (g')^2 + \cancel{x} + xg' - \cancel{x} = 0$

$g'' + 1 + xg' + (g')^2 = 0$

$g' = -\frac{1}{x}$

$S' = 1 - \frac{1}{x}$

$S = x - \ln x \quad \boxed{y = e^{x - \ln x} = \frac{1}{x}e^x}$

b. Write $y = e^S W(x)$, and plug into the differential equation, finding a second order differential equation for W. Attempt a series for $W = \sum_{n=0}^{\infty} a_n x^{-n}$

Classify the $x=\infty$ point using $t=\frac{1}{x}$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2+\frac{dy}{dt}\frac{d^2t}{dx^2}= t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}$$

$$\Rightarrow \quad t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}-t\frac{dy}{dt}-\frac{1}{t}y=0$$

$t=0$ is Irregular singular point

$\rightarrow$ Asymptotic series at $x=\infty$