Jan 1995 #3 (QM)

$V = \frac{1}{2} m\omega^2 x^2$ (Harmonic Oscillator)

$H^1 = \lambda \delta(x - ct)$

$P_{0\to 1}$ from $t = -\infty$ to ∞? $\quad P = |d_1|^2$

$$d_1 = \frac{-i}{\hbar} \int_{-\infty}^{\infty} \langle 1 | H'(t') | 0 \rangle e^{i\omega t'} dt'$$ time dependent perturbation transition formula

$$\langle 1 | H'(t) | 0 \rangle = \int_{-\infty}^{\infty} dx \; u_1^*(x) \, \lambda \delta(x - ct) \, u_0(x)$$

$$u_0(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \exp\left(-\frac{x^2}{2x_0^2}\right) \qquad u_1(x) = \sqrt{2} \, \frac{x}{x_0} u_0(x) \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\langle 1 | H'(t) | 0 \rangle = \frac{\sqrt{2}}{(\pi x_0^2)^{1/2}} \cdot \frac{\lambda}{x_0} \cdot \int_{-\infty}^{\infty} dx \; x \exp\left(-\frac{x^2}{x_0^2}\right) \delta(x - ct)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\lambda c}{x_0^2} \; t \exp\left(-\frac{c^2 t^2}{x_0^2}\right)$$

$$d_1 = \frac{-i}{\hbar} \sqrt{\frac{2}{\pi}} \frac{\lambda c}{x_0^2} \int_{-\infty}^{\infty} dt' \, t' \exp\left(\frac{-c^2 t'^2}{x_0^2} + i\omega t'\right)$$

$$\frac{-c^2 t'^2}{x_0^2} + i\omega t' = \frac{-c^2}{x_0^2}\left(t'^2 - \frac{i\omega x_0^2}{c^2} t'\right) = \frac{-c^2}{x_0^2}\left[\left(t' - \frac{i\omega x_0^2}{2c^2}\right)^2 + \frac{\omega^2 x_0^4}{4c^4}\right]$$

Integral becomes $\displaystyle\int_{-\infty}^{\infty} dt' \, t' \exp\left(\frac{-c^2}{x_0^2}\left(t' - \frac{i\omega x_0^2}{2c^2}\right)^2\right) \exp\left(\frac{-\omega^2 x_0^2}{4c^2}\right)$

$$u = t' - \frac{i\omega x_0^2}{2c^2}$$

$$\exp\left(\frac{-\omega^2 x_0^2}{4c^2}\right) \cdot \int_{-\infty}^{\infty} du \left(\overset{0 \text{ due to oddness}}{\cancel{u}} + \frac{i\omega x_0^2}{2c^2}\right) \exp\left(-\frac{c^2 u^2}{x_0^2}\right)$$

$$= \exp\left(\frac{-\omega^2 x_0^2}{4c^2}\right) \cdot \frac{i\omega x_0^2}{2c^2} \cdot \sqrt{\pi} \cdot \frac{x_0}{c} = \frac{i\omega x_0^3}{2c^3} \sqrt{\pi} \exp\left(\frac{-\omega^2 x_0^2}{4c^2}\right)$$

$$d_1 = \frac{\omega}{\sqrt{2}\hbar}\frac{x_0}{c^2}\lambda \exp\left(\frac{-\omega^2 x_0^2}{4c^2}\right) = \frac{\lambda}{c^2}\sqrt{\frac{\omega}{2\hbar m}} \exp\left(-\frac{\hbar\omega}{4mc^2}\right)$$

$$P_{0\to 1} = |d_1|^2 = \frac{\lambda^2}{2c^4}\frac{\omega}{\hbar m}\exp\left(\frac{-\hbar\omega}{2mc^2}\right)$$