2012, Part 1, Question 3 (jburby@princeton.edu)

(a) For nominal trapping,

$$W_{\perp o} + W_{\parallel o} = W_{\perp o} R,$$

which simply states that all of the ion's energy is perpendicular at the turning point, and that this energy is equal to the ion's initial energy. Because an ion with slightly larger initial parallel energy would leave the device, the trapping condition is

$$W_{\parallel 0} < W_{\perp o}(R-1). \qquad \text{(for trapping)}$$

(b) No sketch.

(c) A turning point z_T will satisfy

$$W_{\perp o} + W_{\parallel o} = \mu B(z_T).$$

Because any turning point must occur at a value of z inside the mirror, i.e. $|z| < cL$, $B(z_T) = B_o(1 + z_T^n/L^n)$. This implies

$$\frac{z^n}{L^n} = \frac{W_{\parallel o}}{W_{\perp o}}.$$

Note that this is not precisely the result the question asks you to find unless $n = 2$. This was probably a typing mistake on part of the question writer.

(d) Because magnetic moment will be conserved, $W_{\perp o}/B_o = W'_{\perp o}/(\beta B_o)$. Thus

$$W'_{\perp o} = \beta W_{\perp o}.$$

(e) A good way to find how the parallel energy changes is to invoke the adiabatic invariance of

$$J = \int_{-z_T}^{z_T} v_\parallel(z) dz,$$

or a constant multiple thereof. A quick calculation shows that, up to a constant of proportionality,

$$v_\parallel(z) = W_{\parallel o}^{1/2}\left(1 - \frac{W_{\perp o}}{W_{\parallel o}}\frac{z^n}{L^n}\right)^{1/2}.$$

Therefore

$$\begin{aligned} J =& W_{\parallel o}^{1/2}\int_{-z_T}^{z_T}\left(1 - \frac{W_{\perp o}}{W_{\parallel o}}\frac{z^n}{L^n}\right)^{1/2} dz \\ =& W_{\parallel o}^{1/2}\frac{W_{\parallel o}^{1/n} L}{W_{\perp o}^{1/n}}\int_{-1}^{1}\frac{u^{\frac{1}{n}-1}}{n}\left(1-u\right)^{1/2} du. \end{aligned}$$

It follows that

$$\frac{W_{\parallel o}^{\frac{n+2}{2n}} L}{W_{\perp o}^{\frac{1}{n}}} = \frac{(W'_{\parallel o})^{\frac{n+2}{2n}}\alpha L}{\beta^{\frac{1}{n}} W_{\perp o}^{\frac{1}{n}}}$$

and

$$W'_{\|o} = \left(\frac{\beta^{\frac{1}{n}}}{\alpha}\right)^{\frac{2n}{n+2}} W_{\|o}.$$

(f) An initially trapped ion will be lost after the change of parameters if its new parallel and perpendicular midplane energies satisfy the loss condition

$$W'_{\|o} > W'_{\perp o}(R-1).$$

Using parts (d) and (e), the latter becomes

$$W_{\|o} > (\beta\alpha^2)^{\frac{n}{n+2}} W_{\perp o}(R-1).$$

So for an initially trapped particle to become lost, the conditions

$$\begin{aligned} &W_{\|o} < W_{\perp o}(R-1) && \text{(initially trapped)} \\ &W_{\|o} > (\beta\alpha^2)^{\frac{n}{n+2}} W_{\perp o}(R-1) && \text{(eventually lost)} \end{aligned}$$

must each be satisfied. Note that in order for there to be any points at all in the energy plane that satisfy these two conditions, it is required that $\beta\alpha^2 < 1$.

(g) No sketch.

(h) The stated condition on α and β would be sufficient to guarantee some amount of detrapping if $\alpha < 1$. However, if $\alpha \geq 1$, this does not seem to be the case. For instance, the limiting case $\beta = \alpha^{1/n}$ appears to maintain all initially trapped particles provided $\beta\alpha^2$ never dipped below 1 during the course of the parametric change.