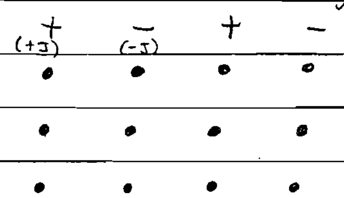
May 1997 #3 (SM)

2D Ising Model, spins are ± 1 on infinite square lattice

Horizontal bonds: exchange constant = J $(J>0)$

Vertical bonds: exchange constant = $\pm J$ (alternates in columns)



By symmetry arguments (translation vertically one row, and translation horizontally two rows), every spin in all the + columns are equivalent, and similarly for − columns.

Consider a given spin in a + column (and a given spin in a − column)

$$H_j^+ = \underbrace{-2JS_{jz}\sum_{k=1}^{2} S_{kz}}_{\text{horizontal interaction}} - \underbrace{2JS_{jz}\sum_{k=1}^{2} S_{kz}}_{\text{vertical interaction}}$$

these spins are in a − column

these spins are in a + column

• Treating only nearest neighbor interactions

$$H_j^- = -2JS_{jz}\sum_{k=1}^{2} S_{kz} + 2JS_{jz}\sum_{k=1}^{2} S_{kz}$$

• Replace + spins by their mean, $\bar{S}_{+z}$, − spins by their mean, $\bar{S}_{-z}$

$$\Rightarrow H_j^+ = -4JS_{jz}\bar{S}_{-z} - 4JS_{jz}\bar{S}_{+z} = -4J(\bar{S}_{+z}+\bar{S}_{-z})S_{jz}$$

$$H_j^- = -4JS_{jz}\bar{S}_{+z} + 4JS_{jz}\bar{S}_{-z} = -4J(\bar{S}_{+z}-\bar{S}_{-z})S_{jz}$$

Solution: (just like paramagnetic spin in a given applied field)

H^+: energy levels = $-4J(\bar{S}_{+z}+\bar{S}_{-z})m_s$ $\quad m_s = -1, 1$ (in this case)

H^-: ....

mean spin given by: $\bar{S}_{jz+} = \tanh(\eta^+)$

$\bar{S}_{jz-} = \tanh(\eta^-)$

where $\eta^+ = \dfrac{4J(\bar{S}_{+z}+\bar{S}_{-z})}{kT}$ $\quad \eta^- = \dfrac{4J(\bar{S}_{+z}-\bar{S}_{-z})}{kT}$

For self consistency in the solution, $\bar{S}_{jz+}$ should equal $\bar{S}_{+z}$, the initial assumed mean spin, and similarly for $\bar{S}_{jz-}$

$\bar{S}_{+z} = \tanh(\eta^+)$ $\quad\quad$ $\bar{S}_{-z} = \tanh(\eta^-)$

Solve for $\bar{S}_{\pm z}$ in terms of $\eta^\pm$:

$$\bar{S}_{+z} = \frac{kT}{8J}(\eta^+ + \eta^-) \quad\quad \bar{S}_{-z} = \frac{kT}{8J}(\eta^+ - \eta^-)$$

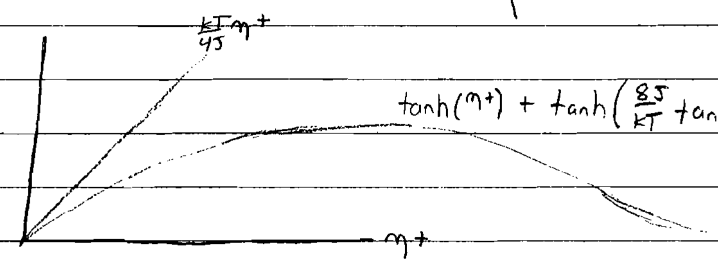
$$\Rightarrow \quad \frac{kT}{8J}(\eta^+ + \eta^-) = \tanh(\eta^+)$$

$$\frac{kT}{8J}(\eta^+ - \eta^-) = \tanh(\eta^-)$$

'put in terms of one variable, η^+,

$$\eta^- = \frac{8J}{kT}\tanh(\eta^+) - \eta^+$$

$$\Rightarrow \quad \frac{kT}{4J}\eta^+ = \tanh(\eta^+) + \tanh\left(\frac{8J}{kT}\tanh(\eta^+) - \eta^+\right)$$



Below $T = T_c$, Ferromagnetic behavior occurs where there is a nonzero possible value for η^+ (and hence $\bar{S}_{+z} + \bar{S}_{-z}$) at zero applied field.

Above $T = T_c$, no intersection point, and $\eta^+ = 0$ is the only solution

$T = T_c$ occurs at small η^+, when slopes of curves are equal

$$\frac{kT_c}{4J}\eta^+ = \eta^+ + \frac{8J}{kT_c}\eta^+ - \eta^+ \quad\quad (\text{small } \eta^+)$$

$$\frac{kT_c}{4J} = \frac{8J}{kT_c}$$

$$(kT_c)^2 = 32J$$

$$\boxed{kT_c = 4\sqrt{2}\,J}$$